

STRUCTURE THEORY OF FLIP GRAPHS WITH APPLICATIONS TO WEAK SYMMETRY BREAKING

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ABSTRACT. This paper is devoted to advancing the theoretical understanding of the iterated immediate snapshot (IIS) complexity of the Weak Symmetry Breaking task (WSB). Our rather unexpected main theorem states that there exist infinitely many values of n , such that WSB for n processes is solvable by a certain explicitly constructed 3-round IIS protocol. In particular, the minimal number of rounds, which an IIS protocol needs in order to solve the WSB task, does not go to infinity, when the number of processes goes to infinity. Our methods can also be used to generate such values of n .

We phrase our proofs in combinatorial language, while avoiding using topology. To this end, we study a certain class of graphs, which we call flip graphs. These graphs encode adjacency structure in certain subcomplexes of iterated standard chromatic subdivisions of a simplex. While keeping the geometric background in mind for an additional intuition, we develop the structure theory of matchings in flip graphs in a purely combinatorial way. Our bound for the IIS complexity is then a corollary of this general theory.

As an afterthought of our result, we suggest to change the overall paradigm. Specifically, we think, that the bounds on the IIS complexity of solving WSB for n processes should be formulated in terms of the size of the solutions of the associated Diophantine equation, rather than in terms of the value n itself.

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1. INTRODUCTION

1.1. Solvability of Weak Symmetry Breaking. The Weak Symmetry Breaking task (WSB) for n processes is an inputless task with possible outputs 0 and 1. A distributed protocol is said to solve the WSB if in any execution without failed processes, there exists at least one process which has value 0 as well as at least one process which has value 1. WSB for n processes is equivalent to the Hard $(2n - 2)$ -Renaming task.

In the classical setting, the processes know their id's, and are allowed to compare them. It is however not allowed that any other information about id's is used. The protocols with this property are called *comparison-based*.¹ In practice this means that behavior of each process only depends on the relative position of its id among the id's of the processes it witnesses and not on its actual numerical value. As a special case, we note that each process must output the same value in case he does not witness other processes at all. Weak Symmetry Breaking is a standard task in theoretical distributed computing, and its solvability is a sophisticated question which has been extensively studied.

The results of this paper apply to the standard computational model called *iterated immediate snapshot*. In this model the processes use two atomic operations being performed on shared memory. These operations are: *write* into the register assigned to that process,

¹Alternative terminology *rank-symmetric* is also used in the literature.

and *snapshot read*, which reads entire memory in one atomic step. Furthermore, it is assumed that the executions are well-structured in the sense that they must satisfy the two following conditions. First, it is only allowed that at each time a group of processes gets active, these processes perform a write operation together, and then they perform a snapshot read operation together; no other interleaving in time of the write and read operations is permitted. Such executions are called *immediate snapshot* executions. Second, each execution can be broken up in rounds, where in every round each non-faulty process gets activated precisely once, alternatively, this can be phrased as each process using fresh memory every time its gets activated.

This model, used extensively in distributed computing, has the major advantage that the protocol complexes have a comparatively simple simplicial structure, and are amenable to mathematical analysis. Specifically, the existence of a distributed protocol solving WSB in r rounds is equivalent to the existence of a certain 0/1-labeling of the vertices of the r th iterated standard chromatic subdivision of an $(n - 1)$ -simplex.

1.2. Previous work. The iterated immediate snapshot model is due to Gafni&Borowsky, see their foundational work [BG93, BG97]; in [HKR] this model goes under the name *layered immediate snapshot*. Several groups of researchers have studied the solvability of the WSB by means of comparison-based IIS protocols. Due primarily to the work of Herlihy&Shavit, [HS], as well as Castañeda&Rajsbaum, [CR08, CR10, CR12a], it is known that the WSB is solvable if and only if the number of processes is not a prime power, see also [AP12] for a counting-based argument for the impossibility part. This makes $n = 6$ the smallest number of processes for which this task is solvable.

The combinatorial structures arising in related questions on subdivisions of simplex paths have been studied in [ACHP13, Ko15b]. The specific case $n = 6$ has been studied in [ACHP13], who has proved the existence of the distributed protocol which solves the WSB task in 17 rounds. This bound was recently improved to 3 rounds in [Ko15a], where also an explicit protocol was given.

We recommend [AW] as a general reference for theoretical distributed computing, and [Ko07] as a general reference for combinatorial topology. Furthermore, our book [HKR] contains all the standard terminology, which we are using here, and which the lack of space does not allow us to introduce in detail.

A broader framework of symmetry breaking tasks can be found in [IRR11]. The topological description for the IIS model can be found in [HKR, HS]; in addition, topological descriptions of several other computational models have also been studied, see [BR15, Ko14a, Ko14b, Ko15a].

1.3. Our results. Our main result states that there is an infinite set of numbers of processes for which WSB can be solved in 3 rounds in the comparison-based IIS model. Specifically, we prove that this is the case when the number of processes is divisible by 6. There is $O(n^2)$ overhead cost to translate an IIS protocol to an IS protocol, so the resulting IS complexity is $O(n^2)$.

Let $sb(n)$ denote the minimal number of rounds which is needed for an IIS protocol to solve WSB for n processes, then our main theorem can succinctly be stated as follows.

Theorem 1.1. *For all $t \geq 1$, we have $sb(6t) \leq 3$.*

Our proof is based on combinatorial analysis of certain matchings in the so-called flip graphs, and strictly speaking does not need any topology.

2. INFORMAL SKETCH OF THE PROOF

2.1. The situation prior to this work. It has long been understood, that there is a 1-to-1 correspondence between IIS protocols solving WSB on one hand and binary assignments λ to the vertices of the iterated chromatic subdivision of a simplex, on the other hand, where these assignments must satisfy certain technical boundary conditions, and have no monochromatic top-dimensional simplices. Furthermore, Herlihy&Shavit found an obstruction to the existence of such an assignment in case the number of vertices of that simplex (i.e., the number of processes) is a prime power. This obstruction is a number which only depends on the values of λ on the boundary of the subdivided simplex, and which must be 0 if there are no monochromatic maximal simplices.

Thus the construction of IIS protocols solving WSB has been reduced to finding such λ , where the number of IIS rounds is equal to the number of iterations of the standard chromatic subdivision. The construction of λ , when the number is not a prime power, was then done by Castañeda&Rajsbaum using the following method. First, boundary values are assigned, making sure that this obstruction value is 0. After that the rest of the values is assigned, taking some care that only few monochromatic maximal simplices appear in the process. After this there is a sophisticated and costly reduction procedure where these monochromatic simplices are connected by paths and eventually eliminated. This elimination procedure is notoriously hard to control, leading to exponential bounds.

2.2. The main ideas of our approach. The idea which we introduce in this paper is radically different. Just as Castañeda&Rajsbaum we produce a boundary labeling making sure the invariant is 0. However after that our approaches diverge in a crucial way. We assign value 0 to **all** internal vertices. This is quite counter-intuitive, as we are trying to get rid of the monochromatic simplices in the long run, while such an assignment on the contrary will produce an enormous amount of them. However, the following key observation comes to our rescue: if we can match the monochromatic simplices with each other so that any pair of matched simplices shares a boundary simplex of one dimension lower, then we can eliminate them all in one go using one more round.

Next, we make a bridge to combinatorics. We have a graph, whose vertices are all the monochromatic maximal simplices, connected by an edge if they share a boundary simplex of one dimension lower; we shall call such graphs *flip* graphs. What we are looking for is a *complete matching* on graphs of this type. This is a simple reduction, but it is very fruitful, since the matching theory on graphs is a very well-developed subject and we find ourselves having many tools at our disposal. A classical method to enlarge existing matchings is that of *augmenting paths*. The idea is elementary but effective: connect the unmatched (also called critical) vertices by a path p , such that all other vertices on the path are matched by the edges along p , and then make all the non-matching edges of p matching and vice versa. This trick will keep all the internal vertices of p matched, while also making end vertices matched. In particular, if we have a matching, and we succeeded to connect critical vertices in pairs by non-intersection augmenting paths, then applying this trick to all these paths simultaneously, we will end up with a complete matching.

2.3. The blueprint of the proof. This set of ideas leads to the following blueprint for constructing the 3-round IIS protocol to solve WSB for n processes:

Step 1. Find a good boundary assignment for the second standard chromatic subdivision of the simplex with n vertices, making sure the Herlihy-Shavit obstruction vanishes. Assign value 0 to *all* internal vertices.

- Step 2.** Decompose the resulting flip graph of monochromatic simplices into pieces corresponding to the maximal simplices of the first chromatic subdivision. Describe an initial matching on each of these pieces, and combine them to a total matching.
- Step 3.** Construct a system of non-intersecting augmenting paths with respect to that total matching. Changing our initial matching along these paths produces the desired complete matching.
- Step 4.** Eliminate all monochromatic maximal simplices in one go, producing a binary assignment for the third standard chromatic subdivision of the simplex with n vertices, which now has no maximal monochromatic simplices.

This is a general scheme, and if the technical details work out, it can be used for various values of n and also for various numbers of rounds. In this paper we restrict ourselves to the values $n = 6, 12, 18, \dots$, mainly because this is the case in which we can provide complete rigorous details. The techniques of this paper can further be extended to discover other values of n for which WSB can be solved in 3 rounds, see [Ko16]. In that paper, progress has been made using techniques of Sperner theory, see [An], specifically a variation of local LYM inequality, to cover values $n = 15, 20, 21, \dots$, see Theorem 12.3.

To start with, for $n = 6t$, there is a quite special arithmetic identity (11.23), which has a stronger set-theoretic version, see the proof of Theorem 1.1, stating the existence of a certain bijection Φ . We produce a quite special labeling of the vertices on the boundary of $\chi^2(\Delta^{n-1})$, and we put the label 0 on all the internal vertices of $\chi^2(\Delta^{n-1})$. Since any top-dimensional simplex has at least one internal vertex, we will have no 1-monochromatic simplices at that point. This corresponds to the step 1 above.

We then proceed with steps 2 and 3, which are at the technical core of our proof. We start with a rough approximation to the matching which we want to get at the end. In this approximation, called the *standard matching*, most of the simplices will get matched. There will be a small number of critical simplices left, concentrated around barycenters of certain boundary simplices. We then find a system of augmenting paths which connect all the critical simplices in pairs. Our idea of how to find these paths is to use the system of non-intersection paths in Γ_n which we construct along the bijection Φ , like a "system of tunnels" between areas of $\chi^2(\Delta^{n-1})$ which contain the monochromatic simplices. Within each such tunnel we use our analysis of the combinatorial structure of the flip graphs, namely certain properties, which we call *conductivity* of these graphs, to connect the critical simplices by augmenting paths, see Figure 6.1. This yields the desired result, allowing us to produce a complete matching on the set of monochromatic simplices. Step 4 is an easy and standard step which has been used before, we do not make any original contribution there.

3. BASIC CONCEPTS

3.1. Graph theory concepts.

We start by recalling some graph terminology, which we need throughout the paper. For a graph G we let $V(G)$ denote the set of its vertices and we let $E(G)$ denote the set of its edges. Two different edges are called *adjacent* if they share a vertex.

Definition 3.1. An *edge coloring* of a graph G with colors from a set C is an assignment $c : E(G) \rightarrow C$, such that adjacent edges get different colors.

Assume G is a graph and A is a subset of $V(G)$. We say that the graph H is the subgraph of G *induced by* A , if the set of vertices of H is A , and two vertices of H are connected by an edge in H if and only if they are connected by an edge in G .

Definition 3.2. A **matching** on a graph G is a set of edges, such that no two of these edges are adjacent. The vertices of these edges are said to be *matched*, while the rest of the vertices are called **critical**.

To underline that not all vertices are matched we often say *partial* matching. On the opposite end, the matching is called *perfect* if all vertices are matched, and it is called *near-perfect* if it has exactly one critical vertex.

3.2. Set theory concepts.

For all natural numbers n , we let $[n]$ denote the set $\{1, \dots, n\}$. Furthermore, throughout the paper we shall skip curly brackets when the set consists of a single element. When we say a *tuple* we mean any ordered sequence. An *order* R on a set S is any tuple $R = (x_1, \dots, x_d)$, satisfying $S = \{x_1, \dots, x_d\}$.

Definition 3.3. Assume A is an arbitrary nonempty finite set of natural numbers, and let k denote the cardinality of A . Then there exists a unique order-preserving bijection $\varphi : A \rightarrow [k]$. We call φ the **normalizer** of A .

Note that if A and B are equicardinal nonempty finite sets of natural numbers, φ is a normalizer of A , and $\gamma : A \rightarrow B$ is the unique order-preserving bijection between A and B , then $\varphi \circ \gamma^{-1}$ is the normalizer of B . On the other hand, if $\psi : B \rightarrow [k]$ is a normalizer of B , then $\psi \circ \gamma$ is a normalizer of A .

When talking about normalizers in the rest of the paper, we shall typically include the information on the set cardinality in normalizers definition. In other words, when we say $\varphi : A \rightarrow [k]$ is a normalizer of A , we mean let k denote the cardinality of A and let the map $\varphi : A \rightarrow [k]$ denote the unique order-preserving bijection.

3.3. S -tuples.

Definition 3.4. For any finite set S , an **S -tuple** is a tuple (A_1, \dots, A_t) of disjoint non-empty subsets of S . We call t the **length** of this S -tuple. We shall use the short-hand notation $A_1 \mid \dots \mid A_t$. An S -tuple $A_1 \mid \dots \mid A_t$ is called **full** if $A_1 \cup \dots \cup A_t = S$. For a full S -tuple $\sigma = A_1 \mid \dots \mid A_t$, we let $V(\sigma)$ denote the set $A_1 \cup \dots \cup A_{t-1} = S \setminus A_t$.

Note, that in [Ko15a] full S -tuples were called *ordered set partitions of the set S* .

Clearly, if $T \supseteq S$, then any S -tuple can be interpreted as a T -tuple as well. Furthermore, if $A = A_1 \mid \dots \mid A_k$ is an S -tuple and B is another S -tuple, we say that A is a *truncation* of B , if B has the form $A_1 \mid \dots \mid A_k \mid A_{k+1} \mid \dots \mid A_t$, for some $t \geq k$.

Distributed Computing Context 3.5. Full $[n]$ -tuples are mathematical objects which encode possible executions of the standard one round protocol for n processes in the immediate snapshot model.

Definition 3.6. Given a set S , and two S -tuples of the same length $\sigma = A_1 \mid \dots \mid A_t$ and $\tau = B_1 \mid \dots \mid B_t$, we call the ordered pair (σ, τ) a **coherent pair of S -tuples**, if we have the set inclusion $B_i \subseteq A_i$, for all $i = 1, \dots, t$.

We find it convenient to view a coherent pair of S -tuples as a $2 \times t$ table of sets

$$(\sigma, \tau) = \begin{array}{|c|c|c|} \hline A_1 & \dots & A_t \\ \hline B_1 & \dots & B_t \\ \hline \end{array}.$$

An arbitrary S -tuple $A_1 \mid \dots \mid A_t$ can alternatively be viewed as a coherent pair of S -tuples $(A_1 \mid \dots \mid A_t, A_1 \mid \dots \mid A_t)$, and we shall use the two interchangeably. We also recall

the following terminology from [Ko15a]: for an S -tuple $\sigma = A_1 | \dots | A_t$ we set its *carrier set* to be $\text{carrier}(\sigma) := A_1 \cup \dots \cup A_t$; for a coherent pair of S -tuples (σ, τ) , we set $\text{carrier}(\sigma, \tau) := \text{carrier}(\sigma)$, and we set $\text{color}(\sigma, \tau) := \text{carrier}(\tau)$; the latter is called the *color set* of (σ, τ) .

Definition 3.7. Assume that we are given a coherent pair of S -tuples, say $(\sigma, \tau) = (A_1 | \dots | A_t, B_1 | \dots | B_t)$, together with a non-empty proper subset $T \subset \text{color}(\sigma, \tau)$. We define a new coherent pair of S -tuples $(\tilde{\sigma}, \tilde{\tau})$, which we call a **restriction** of (σ, τ) to T . To do this, we first decompose $T = T_1 \cup \dots \cup T_d$ as a disjoint union of non-empty subsets such that $T_1 \subseteq B_{i_1}, \dots, T_d \subseteq B_{i_d}$, for some $1 \leq i_1 < \dots < i_d \leq t$. Then, we set $\tilde{A}_1 := A_1 \cup \dots \cup A_{i_1}$, $\tilde{A}_2 = A_{i_1+1} \cup \dots \cup A_{i_2}$, \dots , $\tilde{A}_d = A_{i_{d-1}+1} \cup \dots \cup A_{i_d}$. Finally, we set $\tilde{\sigma} := \tilde{A}_1 | \dots | \tilde{A}_d$, and $\tilde{\tau} := T_1 | \dots | T_d$. We denote this new coherent pair of S -tuples by $(\sigma, \tau) \downarrow T$.

Note, that since an arbitrary S -tuple can be viewed as a coherent pair of S -tuples, we are able to talk about restrictions of S -tuples. However, the set of S -tuples, unlike the set of coherent pairs of S -tuples, is not closed under taking restrictions, and the result of restricting an S -tuple will be a coherent pair of S -tuples.

As an example, let $S = [6]$, $\sigma = \{1, 2\} | 3 | 4 | 5 | 6$, and $T = \{1, 2, 3, 5\}$, then

$$\sigma \downarrow T = \begin{array}{|c|c|c|} \hline 1, 2 & 3 & 4, 5 \\ \hline 1, 2 & 3 & 5 \\ \hline \end{array}.$$

Furthermore, taking $\tilde{T} = \{1, 5\}$, we get

$$\sigma \downarrow \tilde{T} = (\sigma \downarrow T) \downarrow \tilde{T} = \begin{array}{|c|c|} \hline 1, 2 & 3, 4, 5 \\ \hline 1 & 5 \\ \hline \end{array}.$$

Dually, we introduce an operation of *deletion* by setting $\text{dl}((\sigma, \tau), T) := (\sigma, \tau) \downarrow (S \setminus T)$, for an arbitrary coherent pair of S -tuples (σ, τ) , and an arbitrary non-empty proper subset $T \subset S$. In this case we say that the coherent pair of S -tuples $\text{dl}((\sigma, \tau), T)$ is obtained from (σ, τ) by *deleting* T .

4. FLIP GRAPHS

4.1. The graphs Γ_n .

When $\sigma = A_1 | \dots | A_t$ is a full $[n]$ -tuple, we will use the following short-hand notation:

$$F(\sigma) := \begin{cases} [n] \setminus A_t, & \text{if } |A_t| = 1; \\ [n], & \text{otherwise.} \end{cases}$$

The set $F(\sigma)$ is called the *flippable set* of σ .

Definition 4.1. Assume, we are given a full $[n]$ -tuple $\sigma = A_1 | \dots | A_t$, and $x \in F(\sigma)$. Let k be the index, such that $x \in A_k$. We let $\mathcal{F}(\sigma, x)$ denote the full $[n]$ -tuple obtained by using the following rule.

Case 1. If $|A_k| \geq 2$, then set

$$\mathcal{F}(\sigma, x) := A_1 | \dots | A_{k-1} | x | A_k \setminus x | A_{k+1} | \dots | A_t.$$

Case 2. If $|A_k| = 1$ (that is $A_k = x$), then we must have $k < t$. We set

$$\mathcal{F}(\sigma, x) := A_1 | \dots | A_{k-1} | x \cup A_{k+1} | A_{k+2} | \dots | A_t.$$

Due to background geometric intuition, we think of the process of moving from a full $[n]$ -tuple σ to the full $[n]$ -tuple $\tau = \mathcal{F}(\sigma, x)$ as a *flip*. If the first case of Definition 4.1 is

applicable, we say that τ is obtained from σ by *splitting off* the element x , else we say that τ is obtained from σ by *merging in* the element x .

The operation $\mathcal{F}(-, x)$ behaves as a flip operation is expected to behave. Namely, for any $[n]$ -tuple σ and $x \in F(\sigma)$, we have $x \in F(\mathcal{F}(\sigma, x))$, and importantly

$$(4.1) \quad \mathcal{F}(\mathcal{F}(\sigma, x), x) = \sigma.$$

Proposition 4.2. *Assume σ is a full $[n]$ -tuple and $x \in F(\sigma)$, then we have*

$$(4.2) \quad V(\mathcal{F}(\sigma, x)) \cap ([n] \setminus x) = V(\sigma) \cap ([n] \setminus x),$$

where $V(-)$ is as in Definition 3.4.

Proof. Assume $\sigma = A_1 | \dots | A_t$, and assume $x \in A_k$. Let us consider two cases.

Case 1. Assume that $|A_k| \geq 2$. This corresponds to Case 1 of Definition 4.1. If $k \neq t$, then we simply have $V(\mathcal{F}(\sigma, x)) = V(\sigma)$, and (4.2) holds trivially. If $k = t$, then $V(\mathcal{F}(\sigma, x)) = V(\sigma) \cup x$, and so (4.2) is still valid.

Case 2. Assume now that $A_k = x$, $k < t$. This corresponds to Case 2 of Definition 4.1. If $k < t-1$, then we again have $V(\mathcal{F}(\sigma, x)) = V(\sigma)$, while if $k = t-1$, we have $V(\mathcal{F}(\sigma, x)) \cup x = V(\sigma)$. Either way (4.2) is true. \square

We now define the basic flip graphs.

Definition 4.3. *Let n be any natural number. We define a graph Γ_n as follows. The vertices of Γ_n are indexed by all full $[n]$ -tuples. Two vertices σ and τ are connected by an edge if and only if there exists $x \in F(\sigma)$, such that $\tau = \mathcal{F}(\sigma, x)$.*

We can color the edges of the graph Γ_n by elements of $[n]$. To do this we simply assign color x to the edge connecting full $[n]$ -tuple σ with the full $[n]$ -tuple $\mathcal{F}(\sigma, x)$. It follows from (4.1) that this assignment yields a well-defined *edge coloring* of the graph Γ_n ; meaning that we will assign the same color to an edge independently from which of its endpoints we take as σ , and furthermore, any two edges which share a vertex will get different colors under this assignment.

Combinatorially, the edges of Γ_n are indexed by all coherent pairs of $[n]$ -tuples $(\sigma, \tau) = (A_1 | \dots | A_t, B_1 | \dots | B_t)$, satisfying the conditions: $\text{carrier}(\sigma, \tau) = [n]$, and $|\text{color}(\sigma, \tau)| = n - 1$. In this case we have $B_1 \cup \dots \cup B_t = [n] \setminus x$, where x is the color of that edge. Pick index k such that $x \in A_k$. The vertices adjacent to that edge are $A_1 | \dots | A_{k-1} | x | A_k \setminus x | A_{k+1} | \dots | A_t$ and $A_1 | \dots | A_t$. On the other hand, given a vertex $\sigma = A_1 | \dots | A_t$ of Γ_n , and $x \in F(\sigma)$, the edge with color x which is adjacent to σ is indexed by $\text{dl}(\sigma, x)$.

As an example, the $[3]$ -tuples $\sigma_1 = 1 | 23$ and $\sigma_2 = 1 | 3 | 2$ index vertices of Γ_3 . These vertices are connected by an edge labeled with 3 and indexed by the coherent pair of $[3]$ -tuples $(1 | 23, 1 | 2) = \text{dl}(\sigma_1, 3) = \text{dl}(\sigma_2, 3)$.

Distributed Computing Context 4.4. *Two executions are connected by an edge if and only if there exists exactly one process p which has different views under these executions; all other processes have the same view. The color of that edge is p .*

4.2. The graphs Γ_n^2 .

As our next step, we consider ordered pairs of full $[n]$ -tuples. Given two full $[n]$ -tuples σ and τ , we let $\sigma \parallel \tau$ denote the ordered pair (σ, τ) . We shall also combine this with our previous notations, so if $\sigma = A_1 | \dots | A_t$, and $\tau = B_1 | \dots | B_q$, then $\sigma \parallel \tau = A_1 | \dots | A_t \parallel B_1 | \dots | B_q$.

Definition 4.5. We define $F(\sigma \parallel \tau) := F(\sigma) \cup F(\tau)$. Furthermore, for any $x \in F(\sigma \parallel \tau)$ we define

$$(4.3) \quad \mathcal{F}(\sigma \parallel \tau, x) := \begin{cases} \sigma \parallel \mathcal{F}(\tau, x), & \text{if } x \in F(\tau); \\ \mathcal{F}(\sigma, x) \parallel \tau, & \text{if } x \notin F(\tau), x \in F(\sigma). \end{cases}$$

Note, that since $F(\sigma \parallel \tau) := F(\sigma) \cup F(\tau)$, one of the cases in equation (4.3) must occur. We also remark that $F(A_1 \mid \dots \mid A_t \mid B_1 \mid \dots \mid B_q) = [n]$, unless $A_t = B_q = x$, for some $x \in [n]$, in which case we would have $F(A_1 \mid \dots \mid A_t \mid B_1 \mid \dots \mid B_q) = [n] \setminus x$.

Definition 4.6. Let n be any natural number. We define a graph Γ_n^2 as follows. The vertices of Γ_n^2 are indexed by all ordered pairs of full $[n]$ -tuples. Two vertices $\sigma_1 \parallel \tau_1$ and $\sigma_2 \parallel \tau_2$ are connected by an edge if and only if there exists $x \in F(\sigma_1 \parallel \tau_1)$, such that $\sigma_2 \parallel \tau_2 = \mathcal{F}(\sigma_1 \parallel \tau_1, x)$.

To describe the edges of Γ_n^2 , we extend our \parallel -notation and write $(\sigma, \sigma') \parallel (\tau, \tau')$, to denote an ordered pair of coherent pairs of $[n]$ -tuples, subject to an important additional condition:

$$\text{color}(\sigma, \sigma') = \text{carrier}(\tau, \tau').$$

Now, the edges in Γ_n^2 are of two different types, corresponding to the two cases of (4.3):

- either they are indexed by $\sigma \parallel (\tau, \tau')$, where σ is a full $[n]$ -tuple and (τ, τ') indexes an edge in Γ_n ;
- or they are indexed by $(\sigma, \sigma') \parallel \tau$, where (σ, σ') indexes an edge in Γ_n , and τ is a full color (σ, σ') -tuple.

As an example consider the following vertices of the graph Γ_3^2 : $v_1 = 1 \mid 23 \parallel 1 \mid 2 \mid 3$, $v_2 = 1 \mid 23 \parallel 12 \mid 3$, and $v_3 = 1 \mid 3 \mid 2 \parallel 1 \mid 2 \mid 3$. The vertices v_1 and v_2 are connected by an edge labeled by 1 and indexed by $1 \mid 23 \parallel (12 \mid 3, 2 \mid 3)$. The vertices v_1 and v_3 are connected by an edge labeled by 3 and indexed by $(1 \mid 23, 1 \mid 2) \parallel 1 \mid 2$.

We let $\Gamma_n^2(\sigma)$ denote the subgraph of Γ_n^2 induced by the vertices of the form $\sigma \parallel \tau$. Clearly, mapping $\sigma \parallel \tau$ to τ gives an isomorphism between $\Gamma_n^2(\sigma)$ and Γ_n^2 .

4.3. Higher flip graphs, support and subdivision maps.

Since the graphs Γ_n and Γ_n^2 are by far the main characters of this paper, we have chosen to present them separately and in fine detail. However, the constructions from subsections 4.1 and 4.2 can easily be generalized to define the graphs Γ_n^d , for arbitrary $d \geq 1$. Though we will only need these briefly for $d = 3$, we include the general definitions for completeness. The concepts in this subsection were previously introduced in [Ko15a, Section 3].

Let us fix $d \geq 1$ and consider all d -tuples of full $[n]$ -tuples, $v = \sigma_1 \parallel \dots \parallel \sigma_d$. Definition 4.1 can be generalized to such d -tuples as follows. Given v as above we set

$$F(v) := F(\sigma_1) \cup \dots \cup F(\sigma_d).$$

Effectively this means that $F(v) = [n]$, unless $F(\sigma_1) = \dots = F(\sigma_d) = [n] \setminus p$, for some p , in which case we have $F(v) = [n] \setminus p$.

Definition 4.7. Assume v is a d -tuple of $[n]$ -tuples, and $x \in F(v)$. Let k be the maximal index such that $x \in F(\sigma_k)$, by the definition of $F(v)$, such k must exist. We define $\mathcal{F}(v, x)$ to be the following d -tuple of full $[n]$ -tuples:

$$(4.4) \quad \mathcal{F}(v, x) := (\sigma_1 \parallel \dots \parallel \sigma_{k-1} \parallel \mathcal{F}(\sigma_k, x) \parallel \sigma_{k+1} \parallel \dots \parallel \sigma_d).$$

Again, it is easy to see that for any $x \in F(v)$, we have the identities $F(\mathcal{F}(v, x)) = F(v)$ and

$$(4.5) \quad \mathcal{F}(\mathcal{F}(v, x), x) = v.$$

Definition 4.8. For an arbitrary $d \geq 1$ we define graph Γ_n^d as follows. The vertices of Γ_n^d are indexed by all d -tuples of full $[n]$ -tuples. Two vertices v and w are connected by an edge if and only if there exists $x \in F(v)$, such that $w = \mathcal{F}(v, x)$.

In this context, the graph Γ_n^1 is the same as the graph Γ_n . The edges of Γ_n^d are indexed by all d -tuples of coherent pairs of $[n]$ -tuples, which for some $1 \leq k \leq d$ have the special form $\sigma_1 \parallel \dots \parallel \sigma_{k-1} \parallel (\sigma_k, \sigma'_k) \parallel \sigma_{k+1} \parallel \dots \parallel \sigma_d$, where:

- $\sigma_1, \dots, \sigma_{k-1}$ are full $[n]$ -tuples,
- (σ_k, σ'_k) indexes an edge in Γ_n ,
- $\sigma_{k+1}, \dots, \sigma_d$ are full color (σ_k, σ'_k) -tuples.

We call the graphs Γ_n^d *higher flip graphs*. They are related to each other by means of the so-called *support maps*. Specifically, given $c < d$, the support map carrier_d^c goes from the set of vertices of Γ_n^d to the set of vertices Γ_n^c , it takes the d -tuple $\sigma_1 \parallel \dots \parallel \sigma_d$ to the c -tuple $\sigma_1 \parallel \dots \parallel \sigma_c$. If two vertices v and w of Γ_n^d are connected by an edge, then either $\text{carrier}_d^c(v) = \text{carrier}_d^c(w)$ or vertices $\text{carrier}_d^c(v)$ and $\text{carrier}_d^c(w)$ are connected by an edge in Γ_n^c .

Furthermore, assume we are given an arbitrary vertex of Γ_n^c , $v = \sigma_1 \parallel \dots \parallel \sigma_c$, and an arbitrary number $d > c$. We let $\Gamma_n^d(v)$ denote the subgraph of Γ_n^d induced by all vertices w , for which $\text{carrier}_d^c w = v$. Clearly, we have a graph isomorphism $\varphi : \Gamma_n^d(v) \simeq \Gamma_n^{d-c}$, given by $\varphi(\sigma_1 \parallel \dots \parallel \sigma_c \parallel \tau_1 \parallel \dots \parallel \tau_{d-c}) = \tau_1 \parallel \dots \parallel \tau_{d-c}$.

5. STANDARD MATCHINGS ON GRAPHS $\Gamma_n(\Omega, V)$

5.1. Forbidden sets and graphs $\Gamma_n(V)$.

Definition 5.1. Let n be an arbitrary natural number, and let V be any subset of $[n]$. We let $\Gamma_n(V)$ denote the subgraph of Γ_n induced by the vertices indexed by those full $[n]$ -tuples $A_1 \mid \dots \mid A_t$, for which $A_1 \not\subseteq V$.

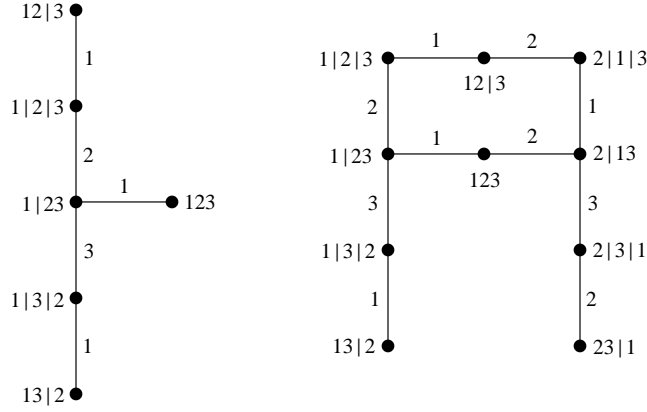
We think of V as a "forbidden set", in which case the condition in Definition 5.1 simply says that the first block of the full $[n]$ -tuple must contain an element which is *not* forbidden. Clearly, when nothing is forbidden, we have no restrictions, hence $\Gamma_n(\emptyset) = \Gamma_n$, and when everything is forbidden, we have no full $[n]$ -tuples satisfying that condition, hence $\Gamma_n([n]) = \emptyset$.

5.2. Prefixes and graphs $\Gamma_n(\Omega, V)$.

Definition 5.2. Let V be any subset of $[n]$, and let $\sigma = A_1 \mid \dots \mid A_t$ be any full $[n]$ -tuple. Let $t-1 \geq k \geq 0$ denote the minimal number for which $A_{k+1} \not\subseteq V$; if no such number exists, we set $k := t$. Then the V -tuple $A_1 \mid \dots \mid A_k$ is called the **V -prefix** of σ , and is denoted by $\text{Pr}_V(\sigma)$.

Note, that we allow the V -prefix of σ to be empty, which is the case if $k = 0$, or equivalently $A_1 \not\subseteq V$, meaning that σ is a vertex of $\Gamma_n(V)$.

Definition 5.3. Let n be an arbitrary natural number, let V be a subset of $[n]$, and let Ω be a family of V -tuples. The graph $\Gamma_n(\Omega, V)$ is the subgraph of Γ_n induced by all vertices σ , such that $\text{Pr}_V(\sigma) \in \Omega$.

FIGURE 5.1. The graphs $\Gamma_3(\{2, 3\})$ (left) and $\Gamma_3(\{3\})$ (right).

Note how this relates to our previously used notations: $\Gamma_n(V) = \Gamma_n(\emptyset, V)$. In line with thinking about the set V as a forbidden set, we think about Ω as a set of *allowed* prefixes.

The next proposition states a simple, but important property of $\text{Pr}_V(\sigma)$.

Proposition 5.4. *Let σ be some full $[n]$ -tuple, and let V be any subset of $[n]$. Assume that we are given $x \in F(\sigma)$, such that $x \notin V$. Then, flipping with respect to x does not change the V -prefix, in other words, we have*

$$(5.1) \quad \text{Pr}_V(\sigma) = \text{Pr}_V(\mathcal{F}(\sigma, x)).$$

Proof. Assume $\sigma = A_1 | \dots | A_k | A_{k+1} | \dots | A_t$, such that $A_1 | \dots | A_k = \text{Pr}_V(\sigma)$, in particular $A_{k+1} \not\subseteq V$. Since $x \notin V$, we have $x \in A_{k+1} \cup \dots \cup A_t$, so we can pick $k+1 \leq l \leq t$, such that $x \in A_l$. We now consider different cases.

First, if $l \geq k+2$, then

$$\mathcal{F}(\sigma, x) = A_1 | \dots | A_k | A_{k+1} | B_{k+2} | \dots | B_{\tilde{t}},$$

for some sets $B_{k+2}, \dots, B_{\tilde{t}}$, where $\tilde{t} = t+1$ or $\tilde{t} = t-1$. Clearly, we then have $\text{Pr}_V(\mathcal{F}(\sigma, x)) = A_1 | \dots | A_k$.

Now assume $l = k+1$ and $|A_{k+1}| = 1$, i.e., $A_{k+1} = x$. Since $x \in F(\sigma)$, we have $k+2 \leq t$. The Case 2 of Definition 4.1 applies, and we have

$$\mathcal{F}(\sigma, x) = A_1 | \dots | A_k | x \cup A_{k+2} | A_{k+3} | \dots | A_t.$$

Hence again $\text{Pr}_V(\mathcal{F}(\sigma, x)) = A_1 | \dots | A_k$, since $x \notin V$.

Finally, assume $l = k+1$, and $|A_{k+1}| \geq 2$. The Case 1 of Definition 4.1 applies, and we have

$$\mathcal{F}(\sigma, x) = A_1 | \dots | A_k | x | A_{k+1} \setminus x | A_{k+2} | \dots | A_t.$$

Since $x \notin V$, we get $\text{Pr}_V(\mathcal{F}(\sigma, x)) = A_1 | \dots | A_k$ here as well. \square

5.3. Standard matchings.

We shall now describe a set of partial matchings on the graphs $\Gamma_n(\Omega, V)$, which we shall call the *standard matchings*. To start with, note that formally a matching is a function μ defined on some of the vertices of G , which has vertices of G as values, and which satisfies the following conditions:

- if $\mu(\sigma)$ is defined, then the vertices σ and $\mu(\sigma)$ are connected by an edge, called the *matching edge*;

- when $\mu(\sigma)$ is defined, then $\mu(\mu(\sigma))$ is also defined and is equal to σ .

When we consider matchings in the specific case of the flip graphs, we can record the labels of the matching edges. Assuming $\mu(\sigma)$ is defined, we let $\text{color}_\mu(\sigma)$ denote the label of the matching edge $(\sigma, \mu(\sigma))$; it is uniquely determined by the identity $\mu(\sigma) = \mathcal{F}(\sigma, \text{color}_\mu(\sigma))$. By (4.1), we have $\text{color}_\mu(\mu(\sigma)) = \text{color}_\mu(\sigma)$.

Definition 5.5. Let V be any subset of $[n]$, and let $R = (x_1, \dots, x_d)$ be an order on its complement $[n] \setminus V$; in particular, $d = n - |V|$. Assume $\sigma = A_1 | \dots | A_t$ is a full $[n]$ -tuple. We set $h_R(\sigma)$ to be the index $1 \leq h \leq d$, such that $\sigma = A_1 | \dots | A_k | x_{h+1} | \dots | x_d$, and $A_k \neq x_h$. Clearly, if such an index exists, it is unique. If it does not exist, we have $\sigma = A_1 | \dots | A_k | x_1 | \dots | x_d$, in which case we set $h_R(\sigma) := 0$. We call $h_R(\sigma)$ the **height** of σ with respect to R .

By Definition 5.5, we have $0 \leq h_R(\sigma) \leq d$, where $d = n - |V|$. The maximum d is achieved if and only if $A_t \neq x_d$. The full $[n]$ -tuples of height 0 with respect to some fixed order R are called *critical with respect to R* .

Remark 5.6. The critical full $[n]$ -tuples all begin by some full V -tuple, followed by the full $([n] \setminus V)$ -tuple $x_1 | \dots | x_d$.

We now have the necessary terminology to define the standard matchings.

Definition 5.7. Assume V is a subset of $[n]$, and R is an order on its complement $[n] \setminus V$. We define a partial matching on the vertices of Γ_n , denoted by μ_R . For an arbitrary full $[n]$ -tuple σ , set $h := h_R(\sigma)$. If $h \neq 0$, we set

$$(5.2) \quad \mu_R(\sigma) := \mathcal{F}(\sigma, x_h),$$

else $\mu_R(\sigma)$ is undefined. We call μ_R the **standard matching** associated to R .

Note that we might as well ask V to be a proper subset of $[n]$ in Definition 5.7. This is because the case $V = [n]$ is rather degenerate: any R is an empty order, and so the standard matching μ_R is empty as well, with all vertices being critical with respect to R .

Proposition 5.8. Whenever V is a subset of $[n]$, and R is any order on $[n] \setminus V$, the partial matching μ_R on the set of vertices of Γ_n is well-defined. Furthermore, when $\mu_R(\sigma)$ is defined, we have

$$(5.3) \quad \text{Pr}_V(\sigma) = \text{Pr}_V(\mu_R(\sigma))$$

and

$$(5.4) \quad h_R(\sigma) = h_R(\mu_R(\sigma)).$$

Proof. To say that μ_R is well-defined is equivalent to the following statements:

- (1) if $\mu_R(\sigma)$ is defined, then σ and $\mu_R(\sigma)$ are connected by an edge;
- (2) $\mu_R(\mu_R(\sigma))$ is also defined;
- (3) $\mu_R(\mu_R(\sigma)) = \sigma$.

The statement (1) is obvious, since $\mu_R(\sigma)$ is a certain flip of σ . To verify the rest, assume $R = (x_1, \dots, x_d)$, and $\sigma = A_1 | \dots | A_k | x_{h+1} | \dots | x_d$, where $h = h_R(\sigma)$. Since $\mu_R(\sigma)$ is defined, we have $h_R(\sigma) > 0$, and $A_k \neq x_h$. By Definition 5.7, we have $\mu_R(\sigma) = \mathcal{F}(\sigma, x_h)$. By construction, we have $x_h \notin V$, so by Proposition 5.4 we get $\text{Pr}_V(\mu_R(\sigma)) = \text{Pr}_V(\mathcal{F}(\sigma, x_h)) = \text{Pr}_V(\sigma)$, and (5.3) is proved.

Pick $1 \leq l \leq k$, such that $x_h \in A_l$. Assume first $|A_l| = 1$, i.e., $A_l = x_h$. In this case we must have $l \leq k - 1$. If $l \leq k - 2$, then

$$\mu_R(\sigma) = A_1 | \dots | A_{l-1} | x_h \cup A_{l+1} | A_{l+2} | \dots | A_k | x_{h+1} | \dots | x_d,$$

and, since $x_h \neq A_k$, we get $h_R(\sigma) = h_R(\mu(\sigma))$. If $l = k - 1$ instead, we have

$$\mu_R(\sigma) = A_1 \mid \dots \mid A_{k-2} \mid x_h \cup A_k \mid x_{h+1} \mid \dots \mid x_d,$$

and, since $x_h \neq x_h \cup A_k$, we again get $h_R(\sigma) = h_R(\mu(\sigma))$.

Assume now that $|A_l| \geq 2$. In this case $\mu_R(\sigma)$ is obtained from σ by splitting off the element x_h . If $l \leq k - 2$, then

$$\mu_R(\sigma) = A_1 \mid \dots \mid A_{l-1} \mid x_h \mid A_l \setminus x_h \mid A_{l+1} \mid \dots \mid A_k \mid x_{h+1} \mid \dots \mid x_d,$$

and, since $x_h \neq A_k$, we get $h_R(\sigma) = h_R(\mu(\sigma))$. Finally, if $l = k - 1$, we have

$$\mu_R(\sigma) = A_1 \mid \dots \mid A_{k-1} \mid x_h \mid A_k \setminus x_h \mid x_{h+1} \mid \dots \mid x_d,$$

and, since $x_h \neq A_k \setminus x_h$, we again get $h_R(\sigma) = h_R(\mu(\sigma))$.

The equality (5.4) has now been proved for all σ . In particular, if $h_R(\sigma) \neq 0$, then $h_R(\mu_R(\sigma)) \neq 0$, so $\mu_R(\mu_R(\sigma))$ is defined.

Finally, the equalities (4.1), (5.4), and definition of μ_R , combine to

$$\mu_R(\mu_R(\sigma)) = \mathcal{F}(\mathcal{F}(\sigma, x_h), x_h) = \sigma,$$

finishing the proof of the proposition. \square

Note that Proposition 5.8, including the equality (5.3), implies that μ_R restricts to a partial matching on $\Gamma_n(\Omega, V)$, for any Ω . The next theorem states the key properties of standard matchings in flip graphs.

Theorem 5.9.

- (1) Let $R = (x_1, \dots, x_n)$ be an arbitrary order on the set $[n]$. The standard matching μ_R on Γ_n has a unique critical vertex, indexed by $x_1 \mid \dots \mid x_n$.
- (2) Let V be an arbitrary non-empty subset of $[n]$, and let R be any order on $[n] \setminus V$. The associated standard matching μ_R on $\Gamma_n(V)$ is complete.
- (3) Assume V is a non-empty subset of $[n]$, $R = (x_1, \dots, x_d)$ an order on $[n] \setminus V$, and Ω a family of V -tuples. The associated standard matching μ_R is a partial matching on $\Gamma_n(\Omega, V)$, with critical vertices of the form $A_1 \mid \dots \mid A_k \mid x_1 \mid \dots \mid x_d$, where $A_1 \mid \dots \mid A_k$ is a full V -tuple in Ω . In particular, if Ω has no full V -tuples, then μ_R is a complete matching on $\Gamma_n(\Omega, V)$.

Proof. (1) We have $V = \emptyset$, hence $\text{Pr}_V(\sigma) = \emptyset$, for all σ . By Remark 5.6 all critical vertices are indexed by concatenations of full V -tuples with $x_1 \mid \dots \mid x_d$, where $d = n - |V|$. Here that description reduces to the existence of a single critical vertex $x_1 \mid \dots \mid x_n$.

(2) Let σ be a vertex of Γ_n which is critical with respect to μ_R . We have $\sigma = A_1 \mid \dots \mid A_k \mid x_1 \mid \dots \mid x_d$, where $A_1 \mid \dots \mid A_k$ is a full V -tuple. Since $V \neq \emptyset$, we have $k \geq 1$ and $A_1 \subseteq V$. By Definition 5.1 this vertex does not belong to $\Gamma_n(V)$, so $\Gamma_n(V)$ has no critical vertices. Clearly, this is the same as to say that μ_R restricts to a complete matching on the vertices of $\Gamma_n(V)$. Identical argument shows the more general statement (3) as well. \square

6. CONDUCTIVITY IN THE FLIP GRAPHS

6.1. Previous work.

While the standard matchings defined in subsection 5.3 are very useful, they do not always yield complete matchings in the situations we will be interested in. It is therefore practical to have a procedure to modify a partial matching so as to decrease the number of critical vertices. To start with, let us recall the following additional terminology from graph theory.

Definition 6.1. Assume we are given a matching on a graph G . An edge path is called **alternating** if its edges are alternating between matching and non-matching ones. It is called **properly alternating** if, in addition, it starts and ends either with a matching edge, or with a critical vertex.

A properly alternating path is called **augmenting** if it starts and ends with critical vertices, it is called **non-augmenting** if it starts and ends with matching edges, and, finally, it is called **semi-augmenting** if it is neither augmenting nor non-augmenting.

Next definition describes a classical technique for modifying matchings.

Definition 6.2. Assume we are given a matching μ on a graph G , and a properly alternating non-self-intersecting edge path γ . We define $D(\mu, \gamma)$ be a new matching on G consisting of all edges from μ which do not belong to γ together with all edges from γ which do not belong to μ .

When trying to modify a matching one is looking for existence of such properly alternating non-self-intersecting edge paths γ . It turns out that when the underlying graph is bipartite, the condition for the path to be non-self-intersecting can be dropped.

Remark 6.3. Assume G is a bipartite graph, μ is a matching on G , v and w are different vertices of G , and γ is a properly alternating edge path from v to w . Then there exists a properly alternating non-self-intersecting edge path from v to w .

Proof. If γ does have self-intersections, then it contains cycles. Any such cycle is of even length, since the graph is bipartite. Deleting a cycle of even length from a properly alternating edge path yields another properly alternating edge path. If we keep removing the cycles, we will eventually make our properly alternating edge path non-self-intersecting. \square

Assume γ is a semi-augmenting path with endpoints v and w , where v is a critical vertex, and w is not. It is easy to see that the set of critical vertices with respect to $D(\mu, \gamma)$ is obtained by taking the critical vertices with respect to μ , and then replacing v with w . For this reason, we shall intuitively view the process of replacing μ with $D(\mu, \gamma)$ as *transporting v to w along the path γ* . We think of the corresponding property of the graph as its *conductivity*. The following result has been proved in [Ko15a].

Proposition 6.4. ([Ko15a, Theorem 5.11, Theorem 5.12])

Assume n is an arbitrary natural number.

- The graph Γ_n is a bipartite graph with a unique bipartite decomposition (A, B) such that $|A| = |B| + 1$. For any vertex $v \in A$ there exists a perfect matching on $\Gamma_n \setminus v$.
- Assume $[n] \supset V \neq \emptyset$, then the graph $\Gamma_n(V)$ is a bipartite graph with a bipartite decomposition (A, B) such that $|A| = |B|$. If furthermore $|V| \leq n - 2$, then for any vertices $v \in A$, $w \in B$, there exists a perfect matching on $\Gamma_n \setminus \{v, w\}$.

The proof of the first part of Proposition 6.4 in [Ko15a] was based on the fact that given a near-perfect matching of Γ_n with a critical vertex $v \in A$, for any other vertex $w \in A$ there would exist a semi-augmenting path from v to w . For the proof of the second part, we constructed in [Ko15a] non-augmenting paths for any pair of vertices $v \in A$ and $w \in B$.

Rather than directly generalizing the techniques used in [Ko15a] to prove Proposition 6.4, we take a slightly different approach. Namely, instead of seeking to connect any pair of arbitrary vertices, we single out a special group of vertices, which we call *connectors* and only try to connect between them.

Definition 6.5. Any vertex of Γ_n , which is indexed by a full $[n]$ -tuple $a_1 | a_2 | \dots | a_n$ is called an a_n -connector of the first type, whereas any vertex indexed by a full $[n]$ -tuple $\{a_1, a_2\} | a_3 | \dots | a_n$ is called an a_n -connector of the second type.

Given a connector $\tau = A_1 | \dots | A_t$ of any of the two types, and a full $[n]$ -tuple σ , we say that τ is **proper** with respect to σ , if $A_1 \notin V(\sigma)$, in other words, τ is a well-defined vertex of $\Gamma_n(V(\sigma))$.

In the rest of this section we assume that $n \geq 5$, and that $n - 1 \geq |V| \geq 1$.

6.2. Conductivity in $\Gamma_n(V, \Omega)$ when Ω has no full V -tuples.

Assume we are given a family of V -tuples Ω which has no full V -tuples. In this case, according to Theorem 5.9(3) the standard matching associated to any order is perfect.

Lemma 6.6. Let $\sigma \in \Gamma_n(V, \Omega)$, $\sigma = a_1 | \dots | a_n$, be an arbitrary a_n -connector of the first type, such that $a_1 \notin V$.

- (1) For any given element $f \neq a_1$, there exists an order R on $[n] \setminus V$, and an edge path p in $\Gamma_n(V, \Omega)$, which is non-augmenting with respect to μ_R , starting from σ , and terminating at some f -connector of the second type $\tau = \{a_1, y_2\} | \dots | y_{n-1} | f$.
- (2) If $|V| \leq n - 2$, there exists an order R on $[n] \setminus V$, and an edge path p in $\Gamma_n(V, \Omega)$, which is non-augmenting with respect to μ_R , starting from σ , and terminating at some a_1 -connector of the second type $\tau = \{y_1, y_2\} | \dots | y_{n-1} | a_1$, with $\{y_1, y_2\} \not\subseteq V$.

Proof. After renaming, we can assume without loss of generality that $V = \{1, \dots, d\}$, where $d \leq n - 1$, and that $a_1 = n$, i.e., $\sigma = n | a_2 | \dots | a_n$. We now set $R := (d + 1, \dots, n)$.

We start by proving (1), i.e., we are given $f \neq a_1$. Assume first that $f = a_l$, for some $l \geq 3$. Using the alternating path swap_k^l shown on left-hand side of the Figure 13.1 we can swap the k -th and the $(k + 1)$ -st parts of our full $[n]$ -tuple, for any $k \geq 3$. We concatenate the paths $\text{swap}_1^l, \text{swap}_{l+1}^l, \dots, \text{swap}_{n-1}^l$ to obtain a new alternating path. This path ends at a vertex of the form $\tilde{\tau} = n | a_2 | b_3 | \dots | b_{n-1} | f$, for some b_3, \dots, b_{n-1} , which is an f -connector of the first type. Set $\tau = \{n, a_2\} | b_3 | \dots | b_{n-1} | f$, and add the matching edge $(\tilde{\tau}, \tau)$ to our path. We now have a non-augmenting path between σ and τ , with the latter being an f -connector of the second type of the required form. Thus the statement (1) is proved in this case.

Assume now that $f = a_2$, i.e., $\sigma = n | f | a_3 | \dots | a_n$. In this case we first follow the somewhat more complicated alternating path swap_2^l shown on the left-hand side of the Figure 13.2, and then proceed as in the case $l \geq 3$, by concatenating the alternating paths $\text{swap}_3^l, \dots, \text{swap}_{n-1}^l$. Again, we will end up with a non-augmenting path between σ and the f -connector of the second type $\tau = \{n, a_3\} | a_4 | \dots | a_n | a_2$. Note, that we use here the fact that $n \geq 5$, implying $n - 1 > 3$. This finishes the proof of (1).

To prove (2) assume now that $|V| \leq n - 2$, in particular, we have $n - 1 \notin V$. Let $l \geq 2$ be such that $a_l = n - 1$. If $l \geq 3$, we start by concatenating the alternating paths $\text{swap}_{l-1}^l, \text{swap}_{l-2}^l, \dots, \text{swap}_2^l$, to arrive at the vertex of the form $n | n - 1 | b_3 | \dots | b_n$, for some b_3, \dots, b_n ; if $l = 2$ then we are at that vertex to start with. Note, that the alternating paths up_k^l , for $1 \leq k \leq n - 1$, allow in certain situations move the element n from being the k -th set of our full $[n]$ -tuple to being its $(k + 1)$ -st set. We now concatenate the alternating paths $\text{up}_1^l, \text{up}_2^l, \dots, \text{up}_{n-1}^l$ to arrive at the vertex $\tilde{\tau} = n - 1 | b_3 | \dots | b_n | n$. To finish, set $\tau = \{n - 1, b_3\} | \dots | b_n | n$, and add the matching edge $(\tilde{\tau}, \tau)$ to our path. We now have a non-augmenting path between σ and τ , where τ is a a_1 -connector of the second type satisfying the desired conditions. This finishes the proof of Lemma 6.6. \square

Lemma 6.7. *Assume we are given a connector of second type $\sigma = \{a_1, a_2\} | a_3 | \dots | a_n$, such that $\{a_1, a_2\} \not\subseteq V$, say $a_1 \notin V$.*

- (1) *For any $f \neq a_1$, there exists an order R on $[n] \setminus V$, and an edge path p in $\Gamma_n(V, \Omega)$, which is non-augmenting with respect to μ_R , starting from σ , and terminating at some f -connector of the first type $\tau = a_1 | y_2 | \dots | y_{n-1} | f$.*
- (2) *If $|V| \leq n - 2$, there exists an order R on $[n] \setminus V$, and an edge path p in $\Gamma_n(V, \Omega)$, which is non-augmenting with respect to μ_R , starting from σ , and terminating at some a_1 -connector of the first type $\tau = y_1 | y_2 | \dots | y_{n-1} | a_1$, with $y_1 \notin V$.*

Proof. Again, we can assume without loss of generality, that $V = \{1, \dots, d\}$, where $d \leq n - 1$, and that $a_1 = n$, i.e., $\sigma = \{n, a_2\} | a_3 | \dots | a_n$. We now set $R := (d + 1, \dots, n)$.

First, we prove the statement (1). Assume $f = a_k$, $k \geq 3$. We can concatenate the paths $\text{swap}_k^{\text{II}}, \dots, \text{swap}_{n-1}^{\text{II}}$, which are shown in the right hand side of Figures 13.1, and 13.2. This will get us to the vertex $\tilde{\tau} = \{n, y_2\} | \dots | y_{n-1} | f$, for some y_2, \dots, y_{n-1} . We set $\tau := n | y_2 | \dots | y_{n-1} | f$ and note that $(\tilde{\tau}, \tau)$ is a matching edge. Adding that edge to the path which we have up to now yields a non-augmenting path connecting σ with τ .

Let us now show (2). We have assumed that $|V| \leq n - 2$, i.e., $n - 1 \notin V$. Here we have $\sigma = \{n, a_2\} | a_3 | \dots | a_n$, and we pick index k such that $a_k = n - 1$. Assume first that $k \geq 3$. If $k = 3$, then we have $\sigma = \{n, a_2\} | n - 1 | a_4 | \dots | a_n$. If $k \geq 4$, then we can concatenate paths $\text{swap}_{k-1}^{\text{II}}, \text{swap}_{k-2}^{\text{II}}, \dots, \text{swap}_3^{\text{II}}$. This will yield an alternating path starting at σ and terminating at $\{n, a_2\} | n - 1 | a_4 | \dots | a_n$. After that we concatenate with the path $\text{specup}^{\text{II}}$ shown on the left hand side of the Figure 13.5. The obtained path terminates at the vertex $\{n - 1, a_2\} | n | a_4 | \dots$. Further, we concatenate with the alternating paths $\text{up}_3^{\text{II}}, \text{up}_4^{\text{II}}, \dots, \text{up}_{n-1}^{\text{II}}$, see the Figure 13.4, to arrive at the vertex $\{n - 1, a_2\} | a_4 | \dots | a_n | n$. We finish by concatenating with the matching edge between $\{n - 1, a_2\} | a_4 | \dots | a_n | n$ and $\tau = n - 1 | a_2 | a_4 | \dots | a_n | n$, to obtain a non-augmenting path from σ to the appropriate n -connector of the first type τ .

It remains to consider the case $k = 2$, that is $\sigma = \{n - 1, n\} | a_3 | a_4 | \dots$. In this situation we start with the alternating path up_2^{II} , see the right hand side of the Figure 13.5, and arrive at the vertex $\{n - 1, a_3\} | n | a_4 | \dots$. We can then proceed just as in the case before with the alternating paths $\text{up}_3^{\text{II}}, \text{up}_4^{\text{II}}, \dots, \text{up}_{n-1}^{\text{II}}$, followed up with the matching edge between $\{n - 1, a_3\} | a_4 | \dots | a_n | n$ and $\tau = n - 1 | a_3 | a_4 | \dots | a_n | n$, to again obtain a non-augmenting path from σ to the appropriate n -connector of the first type τ . This is the last case to be considered and we have now shown the statement (2). \square

Clearly, the Lemmata 6.6 and 6.7 allow us to extend augmenting paths across Γ_n^2 as shown on the Figure 6.1.

Note that if $|V| \leq n - 2$, then there are x -connectors of both types for all x . If $|V| = n - 1$, then there are x -connectors of both types if and only if $x \neq [n] \setminus V$. If $|V| = n - 1$ and $[n] = V \cup \{x\}$, then there are no x -connectors, but we also do not need any.

6.3. Conductivity in $\Gamma_n(V, \Omega)$ in some special cases.

Let us first consider the case when Ω has a unique full V -tuple. Again, according to Theorem 5.9(3) the standard matching associated to any order has a unique critical vertex.

Lemma 6.8. *Assume that the family Ω contains a unique full V -tuple $v_1 | \dots | v_d$ together with all of its truncations. Assume furthermore, that we are given some connector of the first type $\tau = y_1 | \dots | y_n$, such that $y_1 \notin V$. Then there exists an order R on $[n] \setminus V$, and an edge path in $\Gamma_n(V, \Omega)$, which is semi-augmenting with respect to μ_R , and which connects the critical vertex σ to τ .*

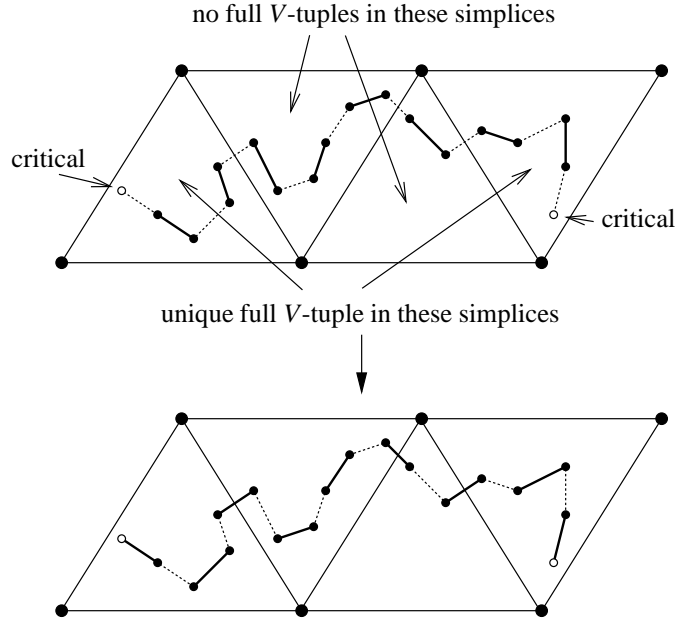


FIGURE 6.1. Concatenating augmenting paths.

Proof. After suitable renaming we can assume, without loss of generality, that $V = \{1, \dots, d\}$, and that the unique full V -tuple is $1 | \dots | d$. Furthermore, we can make sure that $y_1 = n$ after that renaming. We now choose the order $R := (d + 1, \dots, n)$, hence the unique, critical with respect to μ_R , vertex is $\sigma = 1 | 2 | \dots | n$. We need to find a semi-augmenting path from $\tau = n | y_2 | \dots | y_n$ to σ .

To start with, we can concatenate paths swap_k^I , for $2 \leq k \leq n - 1$ in an appropriate order, so as to obtain an alternating path starting at τ and terminating at $n | 1 | 2 | \dots | n - 1$. After this, we concatenate the paths $\text{up}_1^I, \dots, \text{up}_{n-1}^I$. Note, that these paths lie within the graph $\Gamma_n(V, \Omega)$, since we assumed that Ω contains all truncations of $1 | \dots | d$. The total path terminates at σ , which is exactly what we are looking for. \square

Let us now consider the second special case. This time we assume that Ω has three full V -prefixes: $(v_1 | v_2 | v_3 | \dots | v_d)$, $(\{v_1, v_2\} | v_3 | \dots | v_d)$, and $(v_1 | \{v_2, v_3\} | \dots | v_d)$. In this case the standard matching μ_R associated to any order has three critical vertices. We shall extend μ_R by matching two of the critical vertices to each other. After this we find an augmenting path from the third critical vertex to a y -connector, as in Lemma 6.8.

Lemma 6.9. *Assume we are given set V and a family of V -tuples Ω , which contains three full V -tuples as above, together with all of their truncations. Assume, furthermore, we are given $f \in [n]$, such that $[n] \setminus V \neq f$. Then there exists an order R on $[n] \setminus V$, such that the standard matching μ_R can be extended by matching two of the critical vertices to each other, and, furthermore, there exists an edge path in $\Gamma_n(V, \Omega)$, which is semi-augmenting with respect to that extended matching, and which connects the remaining critical vertex σ to some f -connector of the second type $\{y_1, y_2\} | y_3 | \dots | y_{n-1} | f$, such that $\{y_1, y_2\} \not\subseteq V$.*

Proof. Again, after suitable renaming, we can assume, without loss of generality, that $V = \{1, \dots, d\}$, and that the full V -tuples are $(1 | 2 | 3 | \dots | d)$, $(\{1, 2\} | 3 | \dots | d)$, and $(1 | \{2, 3\} | \dots | d)$. Since $[n] \setminus V \neq f$, we can pick an element of $[n] \setminus V$ different from f .

Without loss of generality we can make sure, that after renaming that element is called n . We set $R := (d + 1, \dots, n)$, so the three critical vertices are now $\sigma = \{1, 2\} | 3 | 4 | \dots | n$, $\alpha_1 = 1 | 2 | 3 | 4 | \dots | n$, and $\alpha_2 = 1 | \{2, 3\} | 4 | \dots | n$. We extend the standard matching μ_R by matching α_1 with α_2 .

By our construction, $f \neq n$, and we set $\tau := \{n, 1\} | 2 | \dots | f-1 | f+1 | \dots | n-1 | f$. This is an f -connector of the second type satisfying necessary conditions, since $n \notin V$. We now describe how to find a semi-augmenting path from τ to σ . To start with, we concatenate the paths $\text{swap}_{n-1}^{\text{II}}, \dots, \text{swap}_{f+1}^{\text{II}}$, to arrive at the vertex $\{n, 1\} | 2 | 3 | \dots | n-1$. After this, we concatenate with the path on Figure 13.6 to get to the desired semi-augmenting path to σ . \square

7. NODES

7.1. Definition of n -nodes of the d -th level.

It is now time to define the *nodes*, which, after the flip graphs, constitute the second main combinatorial concept of this paper. On the geometric side the nodes correspond to vertices of iterated chromatic subdivisions, while on distributed computing side they correspond to local views of the processes.

Definition 7.1. Let n and d be arbitrary natural numbers. A d -tuple $v = v_1 \parallel \dots \parallel v_d$ of coherent pairs of $[n]$ -tuples is called an **n -node of the d -th level** if it satisfies the following properties:

- (1) $\text{color}(v_i) = \text{carrier}(v_{i+1})$, for all $1 \leq i \leq d-1$;
- (2) $|\text{color}(v_d)| = 1$, in other words, there exists $S \subseteq [n]$ and $x \in S$, such that $v_d = (S, x)$.

We set $\text{carrier}(v) := \text{carrier}(v_1)$, and call it the **carrier** of v ; we set $\text{color}(v) := \text{color}(v_d)$, and call it the **color** of v .

Let \mathcal{N}_n^d denote the set of all n -nodes of the d -th level. The special cases $d = 1$ and $d = 2$ are the ones most used in this paper, therefore it makes sense to unwind the Definition 7.1 to see explicitly what it says for these values of d .

- An n -node of the first level is simply a pair (S, x) , where $S \subseteq [n]$ and $x \in S$.
- An n -node of the second level is a pair $\sigma \parallel \tau$, where σ is a coherent pair of $[n]$ -tuples, and $\tau = (S, x)$ is an n -node of the first level, such that $\text{color}(\sigma) = S$.

Assume we have a bijective set map $\varphi : S \rightarrow T$. Then we have an induced map taking S -tuples to T -tuples, it is simply given by $\varphi(A_1 | \dots | A_t) = \varphi(A_1) | \dots | \varphi(A_t)$, that is we apply φ to each set separately. In the same way, the function φ extends to coherent pairs of S -tuples, as well as to tuples of coherent pairs of S -tuples.

In particular, assume v is an n -node of d -th level, set $S := \text{carrier}(v)$, and assume we are given a bijective map $\varphi : S \rightarrow T$. Then $\varphi(v)$ is well-defined, it is an n -node of d -th level, and $\text{carrier}(\varphi(v)) = T$.

Definition 7.2. Let v be an n -node of the d -th level, and let φ be the normalizer of $\text{carrier}(v)$. We call $\varphi(v)$ the **normal form** of v .

Definition 7.3. Let n and d be arbitrary natural numbers. Given an n -node of the $(d+1)$ -th level $v = v_1 \parallel \dots \parallel v_{d+1}$, we define a new n -node $w = w_1 \parallel \dots \parallel w_d$ of the d -th level as follows: $w_d := v_d \downarrow \text{color}(v)$, $w_{d-1} := v_{d-1} \downarrow \text{carrier}(w_d)$, \dots , $w_1 := v_1 \downarrow \text{carrier}(w_2)$.

We call the obtained node w the **parent** of v and denote it by $\text{parent}(v)$.

Let us note a few special cases. If $d = 1$, we have $v = v_1 \parallel (S, x)$, and we set $\text{parent}(v) := (T, x) = v_1 \downarrow x$. If $d = 2$, we have $v = v_1 \parallel v_2 \parallel (S, x)$, and we set $\text{parent}(v) := w_1 \parallel (T, x)$, where $(T, x) := v_2 \downarrow x$, and $w_1 := v_1 \downarrow T$.

Distributed Computing Context 7.4. *The nodes are “local views” when n processes run a standard protocol for d rounds.*

7.2. Adjacency of nodes and vertices of the flip graphs.

The n -nodes of the d -th level and vertices of Γ_n^d are related by means of adjacency. We start by giving the general definition.

Definition 7.5. *Let n and d be arbitrary natural numbers. Assume we are given an n -node $v = v_1 \parallel \dots \parallel v_d$ of the d -th level and a vertex $\sigma = \sigma_1 \parallel \dots \parallel \sigma_d$ of Γ_n^d . We say that v and σ are **adjacent** if $v_d = \sigma_d \downarrow \text{color}(v)$ and*

$$(7.1) \quad v_i = \sigma_i \downarrow \text{carrier}(v_{i+1}),$$

for all $i = 1, \dots, d-1$.

It is again instructive to describe explicitly the cases $d = 1$ and $d = 2$. When $d = 1$, we have $v = (S, x)$, and $\sigma = A_1 \mid \dots \mid A_t$. Let k be the index $1 \leq k \leq t$, such that $x \in A_k$. Then the vertex σ and the node (S, x) are *adjacent* if and only if $S = A_1 \cup \dots \cup A_k$.

On the other hand, when $d = 2$, we have an n -node of the second level $v = (\alpha, \beta) \parallel (S, x)$, and a vertex of Γ_n^2 , $\sigma = \sigma_1 \parallel \sigma_2$. Let $\sigma_2 = B_1 \mid \dots \mid B_q$, and let k be the index $1 \leq k \leq q$, such that $x \in B_k$. We say that the vertex σ and the node v are **adjacent** if the following conditions are satisfied:

- $S = B_1 \cup \dots \cup B_k$;
- $(\alpha, \beta) = \sigma_1 \downarrow S$.

It is easy to see that every vertex of Γ_n^d is adjacent to exactly n nodes of d -th level. This is because, once a vertex of Γ_n^d is fixed, the color of the node v defines the node v uniquely by means of equations (7.1).

Distributed Computing Context 7.6. *The adjacency encodes correspondence between local views and global executions.*

7.3. Node labelings.

The main result of this paper is a construction of a function on the set of the nodes satisfying certain constraints.

Definition 7.7. *Assume we are given arbitrary natural numbers n and d . A labeling of the n -nodes of the d -th level, or simply a **node labeling**, is a function $\lambda : \mathcal{N}_n^d \rightarrow \Lambda$, where Λ is an arbitrary set. A **binary node labeling** is a function $\lambda : \mathcal{N}_n^d \rightarrow \{0, 1\}$.*

Unless explicitly stated otherwise, all our node labelings will be binary, so we will frequently omit that word.

Definition 7.8. *An n -node v is called **internal** if $\text{carrier}(v) = [n]$. Any node which is not internal is called a **boundary node**.*

Note that in particular $\text{carrier}(v) \supseteq \text{carrier}(v_i)$, for all $i = 1, \dots, d$, so the carrier of v is sort of a universe, containing all the sets needed to define v .

It is easy to rephrase Definition 7.8 in the special cases $d = 1$ and $d = 2$. An n -node of the first level (S, x) is internal if and only if $S = [n]$, indeed its carrier is simply given by S . On the other hand, an n -node of the second level $(A_1 \mid \dots \mid A_t, B_1 \mid \dots \mid B_r) \parallel (S, x)$ is internal if and only if $A_1 \cup \dots \cup A_t = [n]$.

Definition 7.9. A binary node labeling $\lambda : \mathcal{N}_n^d \rightarrow \{0, 1\}$ is called **blank** if $\lambda(v) = 0$ whenever v is an internal node.

We want to look at the blank binary node labelings which satisfy a certain condition on the boundary.

Definition 7.10. The binary node labeling $\lambda : \mathcal{N}_n^d \rightarrow \{0, 1\}$ is called **compliant** if the following property is satisfied. Assume we are given two n -nodes of the d -th level, say v and w , such that $|\text{carrier}(v)| = |\text{carrier}(w)|$. Let $\varphi : \text{carrier}(v) \rightarrow \text{carrier}(w)$ be the unique order-preserving bijection, and assume furthermore that $w = \varphi(v)$. Then we have $\lambda(v) = \lambda(w)$.

Note, that when $|\text{carrier}(v)| = n$, i.e., when v is an internal node, the condition in Definition 7.10 is empty, since φ must be the identity map. Thus being compliant is really a condition on the boundary nodes in \mathcal{N}_n^d .

Definition 7.11. Assume we are given a binary node labeling $\lambda : \mathcal{N}_n^d \rightarrow \{0, 1\}$. A vertex $\sigma \in V(\Gamma_n^d)$ is called **0-monochromatic** if $\lambda(w) = 0$ for any node $w \in \mathcal{N}_n^d$ which is adjacent to σ . Analogously, a vertex $\sigma \in V(\Gamma_n^d)$ is called **1-monochromatic** if $\lambda(w) = 1$ for any node $w \in \mathcal{N}_n^d$ which is adjacent to σ .

The next definition describes the most important class of node labelings in this paper.

Definition 7.12. A binary node labeling is called **symmetry breaking** if it is compliant and does not have monochromatic vertices.

Proposition 7.13. Any vertex of Γ_n^d is adjacent to some internal node. In particular, Γ_n^d has no 1-monochromatic vertices under a blank binary labeling.

Proof. Assume $\sigma = \sigma_1 \parallel \dots \parallel \sigma_d$ is a vertex of Γ_n^d . Let us say $\sigma_d = A_1 | \dots | A_t$. Take any $x \in A_t$, and let v be the unique n -node of d -th level whose id is x and which is adjacent to σ . It is easy to see, using Definition 7.5, that $v = \sigma_1 \parallel \dots \parallel \sigma_{d-1} \parallel ([n], x)$. Clearly, this node is internal. Finally, this implies that we have no 1-monochromatic vertices, since a blank binary node labeling evaluates to 0 on any internal node. \square

Definition 7.14. Given a binary node labeling $\lambda : \mathcal{N}_n^d \rightarrow \{0, 1\}$, let \mathcal{M}_λ denote the subgraph of Γ_n^d induced by the 0-monochromatic vertices.

Furthermore, for any natural number $q < d$, and whenever σ is a vertex of Γ_n^q , we let $\mathcal{M}_\lambda(\sigma)$ denote the intersection of $\Gamma_n^d(\sigma)$ with \mathcal{M}_λ .

When dealing with blank labelings we shall automatically have no 1-monochromatic vertices. Our next step will be to eliminate 0-monochromatic vertices as well, by looking for matchings on the graph \mathcal{M}_λ .

8. SETS OF PATTERNS

8.1. Definition and some specific sets of patterns.

Definition 8.1. For an arbitrary natural number n , a **set of patterns** in $[n]$ is a union $\mathcal{B}_1 \cup \dots \cup \mathcal{B}_{n-1}$, where for each $1 \leq k \leq n-1$, \mathcal{B}_k is some set of the k -nodes of the first level.

As an example we consider the set of patterns which has been instrumental in our previous work, [Ko15a], when we analyzed the case $n = 6$. Rephrasing the construction from

[Ko15a] in the language of this paper, yields the following set of patterns in [6]:

$$\begin{aligned}\mathcal{B}_1 &= \{(\{1\}, 1)\}, \\ \mathcal{B}_2 &= \{(\{1\}, 1), (\{1, 2\}, 2)\}, \\ \mathcal{B}_3 &= \{(\{1\}, 1), (\{1, 2\}, 1), (\{1, 2\}, 2), (\{1, 2, 3\}, 2), (\{1, 2, 3\}, 3)\}.\end{aligned}$$

The case-by-case analysis which we did in [Ko15a] can be derived from the general structure results which we prove in this paper.

For future reference we define a certain special set of patterns. Assume n is an arbitrary natural number and $\mathbf{x} = (x_1, \dots, x_{n-1})$ is a vector, where $x_1, x_2 \in \{0, 1\}$, $x_i \in \{-1, 0, 1\}$, for all $3 \leq i \leq n-1$. The set of patterns $\mathcal{B}_{\mathbf{x}} = (\mathcal{B}_1, \dots, \mathcal{B}_{n-1})$ is now defined by the following rule:

$$\mathcal{B}_k := \begin{cases} \mathcal{P}_k^+ & \text{if } x_k = 1; \\ \mathcal{P}_k^- & \text{if } x_k = -1; \\ 0 & \text{otherwise;} \end{cases}$$

for all $k = 1, \dots, n-1$, where we set

$$\mathcal{P}_k^+ := \{(\{1\}, 1), (\{1, 2\}, 2), \dots, (\{1, \dots, k\}, k)\}, \text{ for all } 1 \leq k \leq n-1,$$

$$\mathcal{P}_k^- := \mathcal{P}_k^+ \cup \{(\{1, 2\}, 1), (\{1, 2, 3\}, 2)\}, \text{ for all } 3 \leq k \leq n-1.$$

We say that the set of patterns $\mathcal{B}_{\mathbf{x}}$ is *associated* to the vector \mathbf{x} .

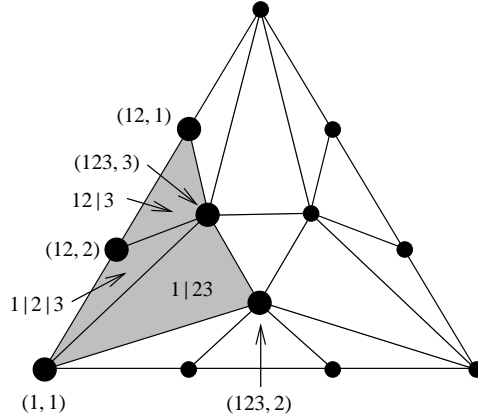


FIGURE 8.1. The set of patterns \mathcal{P}_3^- viewed geometrically.

8.2. Node labeling associated to sets of patterns.

Definition 8.2. Whenever \mathcal{B} is some set of patterns in $[n]$, we define a certain binary node labeling $\lambda_{\mathcal{B}} : \mathcal{N}_n^2 \rightarrow \{0, 1\}$, which we say is **associated** to \mathcal{B} . Pick $v \in \mathcal{N}_n^2$, $v = (A_1 | \dots | A_t, B_1 | \dots | B_t) \parallel (S, x)$. One of the following 3 cases must occur.

Case 1. The n -node v is internal. In that case, we set $\lambda_{\mathcal{B}}(v) := 0$.

Case 2. The n -node v is a boundary node, such that $t \geq 2$. In that case, we set $\lambda_{\mathcal{B}}(v) := 1$.

Case 3. The n -node v is a boundary node, such that $t = 1$, in other words, we can write $v = (A, S) \parallel (S, x)$, where $A \neq [n]$. Let φ be the normalizer of A . We set

$$\lambda_{\mathcal{B}}(v) := \begin{cases} 0, & \text{if } (\varphi(S), \varphi(x)) \in \mathcal{B}; \\ 1, & \text{otherwise.} \end{cases}$$

Definition 8.2 provides us with a method of generating a large family of blank and compliant binary node labelings as the next proposition shows.

Proposition 8.3. *The binary node labeling associated to an arbitrary set of patterns is blank and compliant.*

Proof. Assume \mathcal{B} is some given set of patterns in $[n]$. The binary node labeling $\lambda_{\mathcal{B}}$ is set to be 0 on the internal vertices by definition, so it is blank. To see that it is also compliant, pick two n -nodes of the second level, say v and w , such that $|\text{carrier}(v)| = |\text{carrier}(w)|$. Let $\varphi : \text{carrier}(v) \rightarrow \text{carrier}(w)$ be the unique order-preserving bijection, and assume that $w = \varphi(v)$; in other words, if $v = (A_1 | \dots | A_t, B_1 | \dots | B_t) \parallel (S, x)$, then $w = (\varphi(A_1) | \dots | \varphi(A_t), \varphi(B_1) | \dots | \varphi(B_t)) \parallel (\varphi(S), \varphi(x))$.

If v is an internal vertex, then $v = w$, so $\lambda_{\mathcal{B}}(v) = \lambda_{\mathcal{B}}(w) = 0$. Assume now v is a boundary vertex. If $t \geq 2$, then Case 2 of Definition 8.2 applies both to v and to w , so we get $\lambda_{\mathcal{B}}(v) = \lambda_{\mathcal{B}}(w) = 1$. Assume finally $t = 1$. Let ψ be the normalizer of A_1 , then $\psi \circ \varphi^{-1}$ is a normalizer of $\varphi(A_1)$. Note, that $(\gamma(\varphi(S)), \gamma(\varphi(x))) = (\psi(S), \psi(x))$. In particular $\gamma(\varphi(S), \varphi(x)) \in \mathcal{B}$ if and only if $\psi(S, x) \in \mathcal{B}$, implying that $\lambda_{\mathcal{B}}(v) = \lambda_{\mathcal{B}}(w)$ in this final case as well. \square

Definition 8.4. *Let \mathcal{B} be an arbitrary set of n -nodes of the first level. We let $\mathcal{P}(\mathcal{B})$ denote the set of $[n]$ -tuples $C_1 | \dots | C_t$, such that $(C_1 \cup \dots \cup C_i, x) \in \mathcal{B}$, for any $1 \leq i \leq t$, and any $x \in C_i$. We say that $\mathcal{P}(\mathcal{B})$ consists of all $[n]$ -tuples which can be **composed** from \mathcal{B} .*

Note, that if an $[n]$ -tuple $C_1 | \dots | C_t$ can be composed then then any of its truncations can be composed as well.

Simplicial interpretation 8.5. *The notion of being composed has an interesting simplicial interpretation. Recall, that the n -nodes of first level correspond to vertices of $\chi(\Delta^{n-1})$, which is the standard chromatic subdivision of an $(n-1)$ -simplex. Given \mathcal{B} as in Definition 8.4, we let $K_{\mathcal{B}}$ denote the simplicial complex induced by \mathcal{B} , that is consisting of all simplices from $\chi(\Delta^{n-1})$ whose vertices are in \mathcal{B} . Call a simplex of $\chi(\Delta^{n-1})$ essential if it is contained in a simplex of Δ^{n-1} of the same dimension. Then $\mathcal{P}(\mathcal{B})$ consists of (indexes of) all essential simplices of $K_{\mathcal{B}}$.*

As an example, we have

$$\mathcal{P}(\mathcal{P}_3^-) = \{1, 1 | 2, 12, 1 | 2 | 3, 1 | 23, 12 | 3\}.$$

Simplicially, these correspond to the 6 essential simplices on Figure 8.1: 1 vertex, 2 edges, and 3 triangles.

Definition 8.6. *Assume we are given $\mathcal{B} = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_{t-1}$ - a set of patterns in $[n]$, and $\sigma = A_1 | \dots | A_t$ - a full $[n]$ -tuple. We shall now give an algorithm producing a set of V -tuples, where $V = [n] \setminus A_t$. This set of V -tuples will be denoted $\Omega(\mathcal{B}, \sigma)$.*

Pick some $1 \leq l \leq t-1$. Let $\varphi : A_1 \cup \dots \cup A_l \rightarrow [k]$ be the normalizer of $A_1 \cup \dots \cup A_l$. We define a set of V -tuples Ω_k by saying that a V -tuple $C_1 | \dots | C_q$ belongs to Ω_k if and only if $\varphi(C_1) | \dots | \varphi(C_q)$ belongs to $\mathcal{P}(\mathcal{B}_k)$ and $C_1 \cup \dots \cup C_q \subseteq A_l$. We now set $\Omega(\mathcal{B}, \sigma) := \Omega_1 \cup \dots \cup \Omega_{t-1}$.

Remark 8.7. *Let us note for future reference two important properties of $\Omega(\mathcal{B}, \sigma)$.*

- (1) *The set $\Omega(\mathcal{B}, \sigma)$ is closed under taking truncations.*
- (2) *If $t \geq 3$, then $\Omega(\mathcal{B}, \sigma)$ does not contain any full V -tuples.*

Proof. Assume $C_1 | \dots | C_q \in \Omega(\mathcal{B}, \sigma)$. Picking l as in Definition 8.6 we see that $C_1 \cup \dots \cup C_{q-1} \subseteq A_l$, and $\varphi(C_1) | \dots | \varphi(C_{q-1}) \in \mathcal{P}(\mathcal{B}_k)$; which shows (1). Property (2) follows from the condition that whenever $C_1 | \dots | C_q$ belongs to Ω_k , we have $C_1 \cup \dots \cup C_q \subseteq A_l$, for some $1 \leq l \leq t-1$. \square

8.3. Flip graphs associated to sets of patterns.

The next theorem allows us to understand the combinatorial structure of the subgraphs $\mathcal{M}_{\lambda(\mathcal{B})}(\sigma)$.

Theorem 8.8. *Assume $\mathcal{B} = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_{n-1}$ is an arbitrary set of patterns in $[n]$ and $\sigma = S_1 | \dots | S_t$ is a full $[n]$ -tuple. We have an isomorphism*

$$(8.1) \quad \mathcal{M}_{\lambda_{\mathcal{B}}}(\sigma) \cong \Gamma_n(\Omega(\mathcal{B}, \sigma), V(\sigma)),$$

given by $(\sigma \parallel \tau) \mapsto \tau$.

Proof. Let us take a vertex $\sigma \parallel \tau$ of the graph $\mathcal{M}_{\lambda_{\mathcal{B}}}(\sigma)$ and show that τ is a vertex of $\Gamma_n(\Omega(\mathcal{B}, \sigma), V(\sigma))$. Assume $\tau = T_1 | \dots | T_q$, and choose $1 \leq k \leq q$ to be the minimal index such that $T_{k+1} \cap S_t \neq \emptyset$. We need to show that $T_1 | \dots | T_k \in \Omega(\mathcal{B}, \sigma)$.

Pick an arbitrary $1 \leq i \leq k$, and $x \in T_i$. Set $T := T_1 \cup \dots \cup T_i$. The node of the second level $w = (\sigma \downarrow T) \parallel (T, x)$ is obviously adjacent to the vertex $\sigma \parallel \tau$, so $\lambda_{\mathcal{B}}(w) = 0$. It cannot be internal since $T \cap S_t = \emptyset$, so we assume it is a boundary node. Since $\lambda_{\mathcal{B}}(w) = 0$, and w is a boundary node, we must have $w = (S, T) \parallel (T, x)$, where $S \supseteq T \ni x$. This means that there exists an index $1 \leq d \leq t$, such that $S_d \supseteq T$. Since this is true for any i , we get $S_d \supseteq T_1 \cup \dots \cup T_k$. Furthermore, we have $S = S_1 \cup \dots \cup S_d$, and we set $m := |S|$, and we let $\varphi : S \rightarrow [m]$ to be the normalizer of S . Since $\lambda_{\mathcal{B}}(w) = 0$, we must have $\varphi(T, x) \in \mathcal{B}_m$, for all x , which means precisely that $\varphi(T_1) | \dots | \varphi(T_k) \in \mathcal{P}(\mathcal{B}_k)$. By Definition 8.6 we conclude that $T_1 | \dots | T_k \in \Omega(\mathcal{B}, \sigma)$.

This argument can easily be reversed to show that for any vertex τ in $\Gamma_n(\Omega(\mathcal{B}, \sigma), V(\sigma))$, the vertex $\sigma \parallel \tau$ belongs to $\mathcal{M}_{\lambda_{\mathcal{B}}}(\sigma)$. Finally, we get a graph isomorphism, since graphs on both sides of (8.1) are induced by their respective sets of vertices. \square

9. SETS OF DISJOINT PATHS IN Γ_n

Definition 9.1. A **well-ordered pair** of sets is a pair of sets (S, T) , such that $\emptyset \neq S \subset T \subset [n]$, together with some fixed order on the set T , under which all the elements of S come before all the other elements of T . Two well-ordered pairs of sets (S, T) and (S', T') are called **nested** if either $S \subset S' \subset T' \subset T$ or $S' \subset S \subset T \subset T'$.

For an arbitrary $S \subseteq [n]$, we let \mathbf{b}_S denote the vertex of Γ_n indexed by $S | [n] \setminus S$.

Definition 9.2. Given a well-ordered pair of sets (S, T) , the edge path in Γ_n which is shown on Figure 13.7 connects the vertices \mathbf{b}_S and \mathbf{b}_T . We denote this path by $p_{S,T}$ and call it the **standard path associated to the well-ordered pair (S, T)** .

We shall say that two well-ordered pairs of sets (S, T) and (S', T') are *disjoint* if $\{S, T\} \cap \{S', T'\} = \emptyset$. Clearly, two well-ordered pairs of sets are disjoint if and only if the corresponding paths $p_{S,T}$ and $p_{S',T'}$ have no endpoints in common. The following theorem shows that a much stronger statement is true.

Theorem 9.3. *Assume (S, T) and (S', T') are disjoint well-ordered pairs of sets, which are not nested, then the corresponding standard paths $p_{S,T}$ and $p_{S',T'}$ are disjoint.*

Proof. The informal idea of the proof is that we want to see that one of the endpoints of the standard path is detectable from any vertex on the path. Hence, roughly speaking, if two standard paths have a vertex in common, then they would have to have an endpoint in common.

To start with, we define an operation $ds(-)$. Given a full $[n]$ -tuple $\sigma = A_1 \mid \dots \mid A_t$, pick the indices $1 \leq i_1 < \dots < i_k \leq t$, such that $|A_j| \geq 2$ if and only if $j \in \{i_1, \dots, i_k\}$. We now set

$$ds(\sigma) := A_1 \cup \dots \cup A_{i_1} \mid A_{i_1+1} \cup \dots \cup A_{i_2} \mid \dots \mid A_{i_{k-1}+1} \cup \dots \cup A_{i_k} \mid A_{i_k+1} \cup \dots \cup A_t.$$

In the degenerate case $|A_1| = \dots = |A_t| = 1$, we set $ds(\sigma) := [n]$.

Clearly, $ds(\sigma)$ is again a full $[n]$ -tuple, which either does not have any singletons, or has exactly one singleton as the last set. In general, $\sigma = ds(\sigma)$ if and only if σ either does not have any singletons, or its last set is the only singleton.

Let us now consider a well-ordered pair of sets (S, T) . As the first case we assume that $|S| \geq 2$ and $|T| \geq |S| + 2$. We now apply $ds(-)$ to the vertices of the standard path $p_{S,T}$. The obtained full $[n]$ -tuples are: $(S \mid [n] \setminus S)$, $(T \mid [n] \setminus T)$, and $(S \mid T \setminus S \cup [n] \setminus T)$. In all cases, the first set in that full $[n]$ -tuple is indexing one of the endpoints of $p_{S,T}$, thus if $p_{S,T}$ and $p_{S',T'}$ have a vertex in common, then they also have one of the endpoints in common.

In the remaining cases we still get the same possible patterns for $ds(\sigma)$ with one additional pattern: $ds(\sigma) = [n]$. We get this pattern in two cases:

- when $s = 1$ and $\sigma = x_1 \mid \dots \mid x_k \mid x_{k+1}, \dots, x_t, y_1, \dots, y_{n-t}$ with some $1 \leq k \leq t$;
- when $t = s + 1$ and $\sigma = x_1 \mid \dots \mid x_s \mid x_{s+1} \mid y_1, \dots, y_{n-t}$.

In other words, given σ from $p_{S,T}$ we can always determine either S or T , except for one case. In this case, we have $\sigma = a_1 \mid \dots \mid a_k \mid b_1, \dots, b_{n-k}$. There are two possibilities for the well-ordered pair of sets (S, T) . Either $S = \{a_1\}$ and $\{a_1, \dots, a_k\} \subseteq T$, or $S = \{a_1, \dots, a_{k-1}\}$ and $T = \{a_1, \dots, a_k\}$.

Assume now that the paths $p_{S,T}$ and $p_{S',T'}$ do intersect. Since (S, T) and (S', T') are disjoint, the paths must intersect at an internal point. By what is said above we can assume without loss of generality that $S = \{a_1\}$, $\{a_1, \dots, a_k\} \subseteq T$, $S' = \{a_1, \dots, a_{k-1}\}$, and $T' = \{a_1, \dots, a_k\}$. However, this means that the pairs (S, T) and (S', T') are nested, contradicting our assumptions. \square

10. FROM COMPARABLE MATCHINGS TO SYMMETRY BREAKING LABELINGS

10.1. Complete matchings induce symmetry breaking labelings.

Theorem 10.1. (Theorem A).

Let n be an arbitrary natural number, and let $\lambda : \mathcal{N}_n^2 \rightarrow \{0, 1\}$ be a blank and compliant binary labeling on the n -nodes of second level. Assume that there exists a complete matching on the graph \mathcal{M}_λ , then there exists a symmetry breaking labeling on the n -nodes of third level.

Proof. Let μ denote the complete matching on the graph \mathcal{M}_λ . We now proceed to give a rule defining a binary node labeling $\rho : \mathcal{N}_n^3 \rightarrow \{0, 1\}$ on the n -nodes of the third level. Take $v \in \mathcal{N}_n^3$, $v = v_1 \parallel v_2 \parallel (S, x)$. To start with we set

$$\rho^{\text{def}}(v) := \lambda(\text{parent}(v)),$$

and call this a *default value* of ρ . The rule for defining the value of ρ distinguishes 3 cases.

Case 1. Assume $|S| \leq n - 2$. In this case, we set $\rho(v) := \rho^{\text{def}}(v)$.

Case 2. Assume $|S| = n - 1$. In this case, there exists $y \in [n]$, such that $S = [n] \setminus y$. Since $\text{carrier}(v_2) \supseteq S$, we have $|\text{carrier}(v_2)| \geq n - 1$. If $|\text{carrier}(v_2)| = n - 1$, then we set $\rho(v) := \rho^{\text{def}}(v)$. Else, we must have $\text{carrier}(v_2) = [n]$. Since $\text{color}(v_2) = S$, we see that v_2 is an edge in Γ_n . At the same time v_1 is a full $[n]$ -tuple, so $v_1 \parallel v_2$ is an edge in Γ_n^2 . If the

vertices of Γ_n^2 connected by this edge are matched under μ , then we set $\rho(v) := \rho^{\text{def}}(v)$, else we set $\rho(v) := 1$.

Case 3. Assume $|S| = n$. In other words, we have $S = [n]$. In this case $\alpha = v_1 \parallel v_2$ is a vertex of Γ_n^2 . If this vertex is not monochromatic with respect to λ , then we set $\rho(v) := \rho^{\text{def}}(v)$. If α is monochromatic, then we know that it has been matched, since we assumed that the matching μ is complete. In particular, the label $\text{color}_\mu(\alpha)$ is well-defined. We now complete our definition of ρ by setting

$$\rho(v) := \begin{cases} 0, & \text{if } x = \text{color}_\mu(\alpha); \\ 1, & \text{if } x \neq \text{color}_\mu(\alpha). \end{cases}$$

The value $\rho(v)$ has now been defined for all $v \in \mathcal{N}_n^3$, and we would like to summarize by saying that $\rho(v)$ may be different from the default value $\rho^{\text{def}}(v)$ only in the following two cases:

- if $|S| = n - 1$ and $v_1 \parallel v_2$ is a matching edge;
- if $S = [n]$, $v_1 \parallel v_2$ is a monochromatic vertex, and $x \neq \text{color}_\mu(v_1 \parallel v_2)$.

To see that the node labeling ρ is symmetry breaking, we need to verify that it is compliant, and that it does not have any monochromatic vertices. We start with proving that ρ is compliant. Assume, we have two boundary nodes $v, w \in \mathcal{N}_n^3$, $v = v_1 \parallel v_2 \parallel (S, x)$, $w = w_1 \parallel w_2 \parallel (T, y)$, such that $|\text{carrier}(v)| = |\text{carrier}(w)| \leq n - 1$. By definition of the carrier this means that $|\text{carrier}(v)| = |\text{carrier}(w)|$. Let $\varphi : \text{carrier}(v) \rightarrow \text{carrier}(w)$ be the unique order-preserving bijection. Assume furthermore that $\varphi(v) = w$. Specifically, this means that $w_1 = \varphi(v_1)$, $w_2 = \varphi(v_2)$, $T = \varphi(S)$, and $y = \varphi(x)$.

We now show that under these conditions, we have $\rho(v) = \rho^{\text{def}}(v)$ and $\rho(w) = \rho^{\text{def}}(w)$. First, since $T = \varphi(S)$, we have $|S| = |T|$. Second, we have $S \subseteq \text{carrier}(v)$, so $|S| \leq |\text{carrier}(v)| \leq n - 1$. If $|S| = |T| \leq n - 2$, then $\rho(v) = \rho^{\text{def}}(v)$ and $\rho(w) = \rho^{\text{def}}(w)$ by the Case 1 of our rule. If, on the other hand, $|S| = |T| = n - 1$, then $|\text{carrier}(v)| = |\text{carrier}(w)| = n - 1$, and this time $\rho(v) = \rho^{\text{def}}(v)$ and $\rho(w) = \rho^{\text{def}}(w)$ by the Case 2 of our rule for defining ρ .

Next, let us show that $\varphi(\text{parent}(v)) = \text{parent}(w)$. By the calculation after Definition 7.3, we have $\text{parent}(v) = \gamma_1 \parallel (\tilde{S}, x)$, where $(\tilde{S}, x) = v_2 \downarrow x$, and $\gamma_1 = v_1 \downarrow \tilde{S}$. Clearly, the operation \downarrow commutes with φ , so we have

$$\varphi(\tilde{S}, x) = \varphi(v_2 \downarrow x) = \varphi(v_2) \downarrow \varphi(x) = w_2 \downarrow y$$

and

$$\varphi(\gamma_1) = \varphi(v_1 \downarrow \tilde{S}) = \varphi(v_1) \downarrow \varphi(\tilde{S}) = w_1 \downarrow (w_2 \downarrow y),$$

so

$$\varphi(\gamma_1 \parallel (\tilde{S}, x)) = \varphi(\gamma_1) \parallel (\varphi(\tilde{S}), \varphi(x)) = \text{parent}(w).$$

Let us now verify that ρ has no monochromatic vertices. Let $\sigma = \sigma_1 \parallel \sigma_2 \parallel \sigma_3$ be an arbitrary vertex of Γ_n^3 . We consider two cases.

Case 1. Assume the vertex $\sigma_1 \parallel \sigma_2$ is not monochromatic. We show that $\rho(v) = \rho^{\text{def}}(v)$ whenever v is a node of the third level adjacent to the vertex $\sigma_1 \parallel \sigma_2 \parallel \sigma_3$. Assume $\sigma_3 = A_1 \parallel \dots \parallel A_t$. If $x \in A_k$, such that $|A_1 \cup \dots \cup A_k| \leq n - 2$, then $\rho(v) = \rho^{\text{def}}(v)$ by definition. If, on the other hand, $x \in A_t$, then $\sigma \downarrow x = \sigma_1 \parallel \sigma_2 \parallel ([n], x)$. Since the vertex $\sigma_1 \parallel \sigma_2$ is not monochromatic, we again get $\rho(v) = \rho^{\text{def}}(v)$. The last remaining case is when $x \in A_k$, such that $|A_1 \cup \dots \cup A_k| = n - 1$. This is only possible if $\sigma_3 = A_1 \parallel \dots \parallel A_{t-1} \parallel y$, and $x \in A_{t-1}$. If that happens we have $\sigma \downarrow x = \tau_1 \parallel \tau_2 \parallel ([n] \setminus y, x)$. If now $\tau_1 \parallel \tau_2$ is not an edge, we must have $\rho(v) = \rho^{\text{def}}(v)$. Finally, if $\tau_1 \parallel \tau_2$ is an edge, then $\sigma_1 \parallel \sigma_2$ is one of its

endpoints. However, the vertex $\sigma_1 \parallel \sigma_2$ is not monochromatic by our assumption, and so $\tau_1 \parallel \tau_2$ cannot be a matching edge; hence again we get $\rho(v) = \rho^{\text{def}}(v)$.

We have now proved that $\rho(v) = \rho^{\text{def}}(v)$ whenever v is a node of the third level adjacent to the vertex $\sigma_1 \parallel \sigma_2 \parallel \sigma_3$. On the other hand, the set of parents of these nodes is precisely the set of nodes of the second level adjacent to $\sigma_1 \parallel \sigma_2$, because $\text{parent}(\sigma \downarrow x) = (\sigma_1 \parallel \sigma_2) \downarrow x$. Since we assumed that the vertex $\sigma_1 \parallel \sigma_2$ is not monochromatic, we conclude that neither is the vertex σ .

Case 2. Assume the vertex $\sigma_1 \parallel \sigma_2$ is monochromatic. Since μ is a complete matching, there exists an edge in Γ_n^2 matching $\sigma_1 \parallel \sigma_2$ to some other monochromatic vertex. Set c to be the label of that edge, and assume $\sigma_3 = A_1 \mid \dots \mid A_t$. We now distinguish three further subcases.

Case 2a. Assume $|A_t| \geq 2$. If $x \in A_t$, then set $v_x := \sigma \downarrow x = \sigma_1 \parallel \sigma_2 \parallel ([n], x)$. This is the node with color x adjacent to σ . By our rule, if $x \neq c$, then $\rho(v_x) = 1$. Since $|A_t| \geq 2$, there exists $x \in A_t$ such that $x \neq c$, so at least one of the nodes adjacent to σ has label 1. On the other hand, let v_c be the node of color c adjacent to σ . If $c \in A_t$, then this node has label 0. Otherwise, we have $c \in A_k$, for some $1 \leq k < t$. In this case, we have $v_c := \sigma \downarrow c = \tilde{\sigma}_1 \parallel \tilde{\sigma}_2 \parallel (A_1 \cup \dots \cup A_k, c)$. Since $|A_1 \cup \dots \cup A_k| \leq n - 2$, we get $\rho(v_c) = 0$ again. Either way, we have nodes with different labels adjacent to σ , so σ is not monochromatic.

Case 2b. Assume $|A_t| = 1$, $A_t = \{y\}$, with $y \neq c$. As above we calculate $\rho(v_y) = 1$. Take now $x \in A_{t-1}$. If $\rho(x) = \rho^{\text{def}}(x)$, then $\rho(x) = 0$, since the vertex $\sigma_1 \parallel \sigma_2$ is 0-monochromatic. Otherwise, we have $\sigma \downarrow x = \tilde{\sigma}_1 \parallel \tilde{\sigma}_2 \parallel ([n] \setminus y, x)$, and $\tilde{\sigma}_1 \parallel \tilde{\sigma}_2$ is a matching edge. This is impossible, since that edge would be labeled y , and have $\sigma_1 \parallel \sigma_2$ as one of its endpoints, contradicting the assumption $y \neq c$. In either case we have two nodes adjacent to σ with different values of ρ , so σ is not monochromatic.

Case 2c. Assume $|A_t| = 1$, $A_t = \{c\}$. By the calculation in Case 2a, we get $\rho(v_c) = 0$. Take any $x \in A_{t-1}$, so $v_x = \sigma \downarrow x = \tilde{\sigma}_1 \parallel \tilde{\sigma}_2 \parallel ([n] \setminus c, x)$. Now $\tilde{\sigma}_1 \parallel \tilde{\sigma}_2$ is an edge labeled c which has $\sigma_1 \parallel \sigma_2$ as an endpoint. By assumptions above this means that $\tilde{\sigma}_1 \parallel \tilde{\sigma}_2$ is a matching edge, and so we get $\rho(v_x) = 1$. Again, we have different values of ρ on the nodes adjacent to σ , so σ is not monochromatic.

This finishes the proof of the theorem. \square

Corollary 10.2. *Let n be an arbitrary natural number, and let \mathcal{B} be an arbitrary set of patterns in $[n]$. If there exists a complete matching on the graph $\mathcal{M}_{\lambda_{\mathcal{B}}}$, then there exists a symmetry breaking labeling on the n -nodes of third level.*

Proof. The binary node labeling $\lambda_{\mathcal{B}}$ associated to the set of patterns \mathcal{B} is always blank and compliant, see Proposition 8.3, hence the statement follows from Theorem 10.1. \square

10.2. Non-intersecting path systems induce complete matchings.

Definition 10.3. *Let n be an arbitrary natural number, $n \geq 2$. The linear Diophantine equation in $n - 1$ variables*

$$(10.1) \quad x_1 \binom{n}{1} + x_2 \binom{n}{2} + \dots + x_{n-1} \binom{n}{n-1} = 1$$

*is called **binomial Diophantine equation** associated to n .*

A solution (x_1, \dots, x_{n-1}) to the binomial Diophantine equation associated to n is called *primitive* if $x_1 = 1$, $x_2 \in \{0, 1\}$, and $x_i \in \{-1, 0, 1\}$, for all $i = 3, \dots, n-1$. For example $(1, 1, -1, 0, 0)$ is a primitive solution for $n = 6$, and $(1, 0, -1, 1, 0, 0, 0, -1, -1, 0)$ is a primitive solution for $n = 12$.

We shall now consider families of proper subsets of the set $[n]$, i.e., $\Sigma \subseteq 2^{[n]} \setminus \{\emptyset, [n]\}$, which we call *proper families*. Let C_t^n be the family of all subsets of $[n]$ of cardinality t . It is a proper family if $1 \leq t \leq n-1$.

Definition 10.4. A proper family $\Sigma \subseteq 2^{[n]}$ is called **cardinal** if the following is satisfied: whenever $S \in \Sigma$, we have $C_{|S|}^n \subseteq \Sigma$.

A cardinal family Σ can be described simply by specifying the list of cardinalities $C(\Sigma) \subseteq \{1, \dots, n-1\}$ of the sets in Σ , namely $\Sigma = \cup_{t \in C(\Sigma)} C_t^n$. Two proper families Σ and Λ are *disjoint* if and only if they are disjoint as sets. Clearly, cardinal families Σ and Λ are disjoint if and only if the corresponding cardinality sets $C(\Sigma)$ and $C(\Lambda)$ are disjoint.

Assume now that we are given $n \geq 2$, and that $\mathbf{x} = (x_1, \dots, x_{n-1})$ is a primitive solution to the binomial Diophantine equation associated to n . Set $I_{\mathbf{x}} := \{i \mid i \in [n], x_i = 1\}$ and $J_{\mathbf{x}} := \{j \mid j \in [n], x_j = -1\}$, in particular $1 \in I_{\mathbf{x}}$. Furthermore, set $\Sigma_{\mathbf{x}} := \{S \mid S \subset [n], |S| \in I_{\mathbf{x}}\}$ and $\Lambda_{\mathbf{x}} := \{T \mid T \subset [n], |T| \in J_{\mathbf{x}}\}$. Clearly, $\Sigma_{\mathbf{x}}$ and $\Lambda_{\mathbf{x}}$ are proper families of subsets. They are disjoint because $I_{\mathbf{x}}$ and $J_{\mathbf{x}}$ are disjoint. Finally, since \mathbf{x} is a primitive solution to the binomial Diophantine equation associated to n we have $|\Sigma_{\mathbf{x}}| = |\Lambda_{\mathbf{x}}| + 1$. We say that the proper families of subsets $\Sigma_{\mathbf{x}}$ and $\Lambda_{\mathbf{x}}$ are *associated to \mathbf{x}* .

Definition 10.5. Assume we are given two proper set families Σ and Λ , such that $|\Sigma| = |\Lambda|$. A **non-intersecting path system** between Σ and Λ consists of a bijection $\varphi : \Sigma \rightarrow \Lambda$ together with a set of disjoint edge paths $\{q_{S, \varphi(S)}\}_{S \in \Sigma}$, such that each path $q_{S, \varphi(S)}$ connects \mathbf{b}_S with $\mathbf{b}_{\varphi(S)}$.

Theorem 10.6. (Theorem B).

Assume $\mathbf{x} = (x_1, \dots, x_{n-1})$ is a primitive solution to the binomial Diophantine equation associated to some $n \geq 2$, and $\Sigma_{\mathbf{x}}$ and $\Lambda_{\mathbf{x}}$ are the associated proper families of subsets of $[n]$. If there exists a non-intersecting path system between $\Sigma_{\mathbf{x}} \setminus \{n\}$ and $\Lambda_{\mathbf{x}}$, then there exists a compliant symmetry breaking binary labeling on the n -nodes of the third level.

Proof. Set as above $I_{\mathbf{x}} := C(\Sigma_{\mathbf{x}})$ and $J_{\mathbf{x}} := C(\Lambda_{\mathbf{x}})$. We have a bijection $\varphi : \Sigma_{\mathbf{x}} \setminus \{n\} \rightarrow \Lambda_{\mathbf{x}}$, and a family of disjoint edge paths in Γ_n , $\{q_{S, \varphi(S)}\}_{S \in \Sigma_{\mathbf{x}} \setminus \{n\}}$, such that each path $q_{S, \varphi(S)}$ connects \mathbf{b}_S with $\mathbf{b}_{\varphi(S)}$. Let $\mathcal{B}_{\mathbf{x}} = (\mathcal{B}_1, \dots, \mathcal{B}_{n-1})$ be the set of patterns associated to vector \mathbf{x} , and consider the associated node labeling $\lambda = \lambda_{\mathcal{B}_{\mathbf{x}}} : \mathcal{N}_n^2 \rightarrow \{0, 1\}$.

We have a bipartite graph \mathcal{M}_{λ} , and we are looking for a complete matching on this graph. This graph consists of subgraphs $\mathcal{M}_{\lambda}(\sigma)$ where $\sigma = A_1 \mid \dots \mid A_t$ ranges through vertices of Γ_n . We can start by taking some matchings on these subgraphs and then eliminating the remaining critical vertices. Theorem 8.8 describes $\mathcal{M}_{\lambda}(\sigma)$ combinatorially, for each σ . All these graphs are isomorphic to $\Gamma_n(\Omega(\mathcal{B}_{\mathbf{x}}, \sigma), V)$. By Remark 8.7, the set $\Omega(\mathcal{B}_{\mathbf{x}}, \sigma)$ does not contain any full V -tuples whenever $t \geq 3$, so in these cases $\mathcal{M}_{\lambda}(\sigma)$ will have a complete matching: we can take the standard matching with respect to any order on $[n] \setminus V$ and then apply Theorem 5.9(3).

When $t = 2$, we have $\sigma = S \mid [n] \setminus S = \mathbf{b}_S$, say $S = \{x_1, \dots, x_k\}$, for $x_1 < \dots < x_k$. In this case the standard matchings on $\mathcal{M}_{\lambda}(\sigma)$ are not complete. They have critical vertices which are in a bijective correspondence with full V -prefixes. Going back to the definition of the set of patterns associated to \mathbf{x} , we distinguish 3 cases.

- Case 1.** If $|S| \in I_x$, then the set $\Omega(\mathcal{B}_x, \sigma)$ has a unique full V -prefix, namely $x_1 | \dots | x_k$. Thus for any order on $[n] \setminus V$ the standard matching will have a unique critical vertex.
- Case 2.** If $|S| \in J_x$, then the set $\Omega(\mathcal{B}_x, \sigma)$ has 3 full V -prefixes, namely $x_1 | \dots | x_k$, $x_1, x_2 | x_3 | \dots | x_k$, and $x_1 | x_2, x_3 | \dots | x_k$. Note, that in this case we must have $k \geq 3$. Thus for any order on $[n] \setminus V$ the standard matching will have the corresponding 3 critical vertices.
- Case 3.** If $|S| \notin I_x \cup J_x$, then $\Omega(\mathcal{B}_x, \sigma)$ is empty, and it follows from Theorem 5.9(2) that the standard matching is complete.

Finally, when $t = 1$, we have $\sigma = [n]$. In this case Theorem 5.9(1) applies and we have a unique critical vertex which depends on the chosen order.

We now use conductivity in flip graphs, as developed in Section 6, to find edge paths in Γ_n^2 connecting all the critical vertices. Let us fix $S \in \Sigma_x \setminus \{n\}$, and take the corresponding path $q_{S, \varphi(S)}$. We shall be traversing that path starting from $\mathbf{b}_{\varphi(S)}$ and going towards \mathbf{b}_S , so we let $w_1 := \mathbf{b}_{\varphi(S)}$, $w_2, \dots, w_{d-1}, w_d := \mathbf{b}_S$ denote the vertices on the path listed in that order. Note that d must be odd. For $k = 1, \dots, d-1$, let y_k be the label of the edge between w_k and w_{k+1} . By Lemma 6.9 one can choose the order R , so that the standard matching can be extended to match two of the critical vertices to each other, and there will exist a semi-augmenting edge path connecting the remaining critical vertex to some y_1 -connector of the second type τ_1^f , which is proper with respect to $\mathbf{b}(\varphi(S))$.

Let τ_2^s be the unique vertex connected to τ_1^f by the edge with label y . Clearly, τ_2^s is a y_1 -connector of the second type, which by the identity (4.2) is proper with respect to w_2 . By Lemma 6.6, there exists an order R and a non-augmenting edge path in the graph $\mathcal{M}_\lambda(w_2)$ with respect to μ_R , connecting τ_2^s to some y_2 -connector of the first type τ_2^f , which is proper with respect to w_2 . We now let τ_3^s be the unique vertex connected to τ_2^f by the edge with the label y_2 , which is a y_2 -connector of the first type proper with respect to w_3 . We then repeat that argument for the graph $\mathcal{M}_\lambda(w_3)$, using Lemma 6.7 instead.

Eventually, we will arrive at a vertex τ_d^s in $\mathcal{M}_\lambda(\mathbf{b}_S)$. Since d is odd, τ_d^s is a y_{d-1} -connector of the first type, and it is proper with respect to \mathbf{b}_S . We now employ Lemma 6.8, which tells us that there exists an order R and a semi-augmenting path with respect to μ_R which connects τ_d^s with the unique critical vertex in $\mathcal{M}_\lambda(\mathbf{b}_S)$. Concatenating all these paths will yield an augmenting path which connects the two critical vertices in $\mathcal{M}_\lambda(\mathbf{b}_{\varphi(S)})$ and $\mathcal{M}_\lambda(\mathbf{b}_S)$. Applying the transformation from Definition 6.2 to that path will yield a new matching, where these two critical vertices are now matched. Doing this for all paths $q_{S, \varphi(S)}$, when S ranges over all subsets in $\Sigma_x \setminus \{n\}$ will yield a matching on \mathcal{M}_λ , with two of the critical vertices remaining: one in $\mathcal{M}_\lambda(n | [n-1])$, and one in $\mathcal{M}_\lambda([n])$.

Note, that by Theorem 8.8, we have $\mathcal{M}_\lambda([n]) \cong \Gamma_n$ and $\mathcal{M}_\lambda(n | [n-1]) \cong \Gamma_n(\Omega, n)$, where $\Omega = \{n\}$. To start with, consider the standard matching in $\mathcal{M}_\lambda([n])$ with respect to the order $(1, \dots, n)$, by Theorem 5.9, we have a unique critical vertex v indexed by $[n] \parallel 1 | 2 | \dots | n$. Let w be the vertex of $\mathcal{M}_\lambda(n | [n-1])$ indexed by $n | [n-1] \parallel 1 | 2 | \dots | n$. Clearly, the vertices v and w are connected in \mathcal{M}_λ by an edge labeled n . By Lemma 6.8, there exists an order on $[n-1]$, such that there exists a semi-augmenting path connecting the unique critical vertex with w . In fact the order $(1, \dots, n-1)$ will do, in which case the unique critical vertex will be $n | [n-1] \parallel n | 1 | \dots | n-1$, and the semi-augmenting path can be given explicitly: $n | 1 | \dots | n-1 \rightarrow 1, n | 2 | \dots | n-1 \rightarrow 1 | n | 2 | \dots | n-1 \rightarrow \dots \rightarrow 1 | 2 | \dots | n-1, n \rightarrow 1 | 2 | \dots | n$. Concatenating this path with the edge between v and w yields an augmenting path which eliminates the last two critical vertices, resulting in a perfect matching on \mathcal{M}_λ . \square

10.3. Comparable matchings induce non-intersection path systems.

Producing a non-intersecting path system for $n = 6$, and $\mathbf{x} = (1, 1, -1, 0, 0)$ has been done by hand in [Ko15a]. Unfortunately, doing it directly appears prohibitive for larger values of n . We now look for further structures which will help us construct non-intersecting path systems.

Definition 10.7. A **comparable matching** between disjoint proper families Σ and Λ is a bijection $\varphi : \Sigma \rightarrow \Lambda$, such that for any $S \in \Sigma$, either $(S, \varphi(S))$ or $(\varphi(S), S)$ is a well-ordered pair. We say that this well-ordered pair is **associated** to S .

The comparable matching φ is called **non-nested** if for any $S, T \in \Sigma$ the associated well-ordered pairs are not nested.

Given disjoint proper families Σ and Λ , the set of comparable matchings $\varphi : \Sigma \rightarrow \Lambda$ can be partially ordered as follows. Assume we have two subsets $S, T \subseteq [n]$, such that one contains the other one. If $S \subset T$, then we set $l(S, T) := |T \setminus S|$, else set $l(S, T) := |S \setminus T|$; this is a distance between S and T . Let L_φ denote the multiset of distances $\{l(S, \varphi(S)) \mid S \in \Sigma\}$. We define an associated function dist_φ on the set of natural numbers, by setting $\text{dist}_\varphi(d)$ to be the number of occurrences of d in L_φ . Since L_φ is a finite multiset, the value $\text{dist}_\varphi(d)$ is different from 0 for only finitely many values of d .

Assume now we are given two comparable matchings $\varphi, \psi : \Sigma \rightarrow \Lambda$. If the functions dist_φ and dist_ψ are identical, we say that φ and ψ are incomparable. Otherwise, let k be the maximal index such that $\text{dist}_\varphi(d) \neq \text{dist}_\psi(d)$. We now say that $\varphi < \psi$ if $\text{dist}_\varphi(d) < \text{dist}_\psi(d)$, and we say that $\varphi > \psi$ if $\text{dist}_\varphi(d) > \text{dist}_\psi(d)$. Clearly, this is a well-defined partial order on the set of all comparable matchings between Σ and Λ , which we call *distance-lexicographic order*.

Proposition 10.8. Assume we have proper families Σ and Λ , and a comparable matching $\varphi : \Sigma \rightarrow \Lambda$, then there exists a non-nested comparable matching $\psi : \Sigma \rightarrow \Lambda$.

Proof. Without loss of generality, we can assume that φ is chosen to be a comparable matching which is minimal with respect to the distance-lexicographic order defined above. If φ is non-nested, then we are done, so assume this is not the case and take any pair $S, T \in \Sigma$ such that the associated well-ordered pairs are nested. Without loss of generality, swapping Σ and Λ if necessary, we can assume that $S \subset \varphi(S)$, $S \subset T$, and $S \subset \varphi(T)$. We then have two cases, either we have $S \subset T \subset \varphi(T) \subset \varphi(S)$, or $S \subset \varphi(T) \subset T \subset \varphi(S)$.

Define a new bijection $\psi : \Sigma \rightarrow \Lambda$ as follows: $\psi(A) := \varphi(A)$, for $A \neq S, T$, $\psi(S) := \varphi(T)$, and $\psi(T) := \varphi(S)$. Clearly, ψ is again a comparable matching, which precedes φ in the distance-lexicographic order. This contradicts the choice of φ . \square

Theorem 10.9. (Theorem C).

Assume $\mathbf{x} = (x_1, \dots, x_{n-1})$ is a primitive solution to the binomial Diophantine equation associated to some $n \geq 2$, and $\Sigma_{\mathbf{x}}$ and $\Lambda_{\mathbf{x}}$ are the associated proper families of subsets of $[n]$. If there exists a comparable matching between $\Sigma_{\mathbf{x}} \setminus \{n\}$ and $\Lambda_{\mathbf{x}}$, then there exists a compliant symmetry breaking binary labeling on the n -nodes of the third level.

Proof. Consider a comparable matching $\varphi : \Sigma_{\mathbf{x}} \setminus \{n\} \rightarrow \Lambda_{\mathbf{x}}$. By Proposition 10.8 we might as well assume that φ is non-nested. By Theorem 9.3 the family $\{p_{S, \varphi(S)}\}_{S \in \Sigma_{\mathbf{x}} \setminus \{n\}}$ is a non-intersecting path system, so the result follows from Theorem 10.6. \square

Distributed Computing Context 10.10. Theorem 10.9 means that in the standard computational model the existence of a comparable matching between disjoint cardinal proper families of subsets of $[n]$ implies the existence of a wait-free protocol solving Weak Symmetry Breaking in 3 rounds.

11. NEW UPPER BOUNDS FOR $\text{sb}(n)$

11.1. The formulation of the main theorem and some set theory notations.

Our goal now is to use Theorem 10.9 to improve upper bounds for the symmetry breaking function $\text{sb}(n)$. Our most definite result is to show that there are infinitely many values of n for which $\text{sb}(n) \leq 3$.

We now return to considering Theorem 1.1. The case $t = 1$ has been previously settled in [Ko15a]. To deal with the case $t = 2$, we first note that we have the following arithmetic identity:

$$\binom{12}{1} + \binom{12}{4} = \binom{12}{0} + \binom{12}{3} + \binom{12}{9} + \binom{12}{10} = 507.$$

We can then use computer search to show the existence of a comparable matching between the corresponding disjoint cardinal proper families of subsets of [12]. Clearly, to deal with the general case, we need to move beyond the direct computer search.

Before proceeding with the proof of Theorem 1.1 we need a little bit of terminology. We shall think about subsets of the set $[n]$ in terms of their support vector, i.e., we identify a subset $S \subseteq [n]$ with an n -tuple $\chi_S = (a_1, \dots, a_n)$, where $a_i = 1$ if $i \in S$ and $a_i = 0$ otherwise.

Definition 11.1. Let α be any tuple of length at most n , consisting of 0's and 1's. We let $\langle \alpha \rangle_n$ denote the set of all subsets S whose support vector S ends with α . If n does not matter, we shall drop it, and simply write $\langle \alpha \rangle$.

So, if $\alpha = (a_1, \dots, a_k)$, then the set $\langle \alpha \rangle$ consists of all subsets S , for which we have $\chi_S = (b_1, \dots, b_{n-k}, a_1, \dots, a_k)$, or, in other words, for $i = n - k + 1, \dots, n$, we have $i \in S$ if and only if $a_{i+k-n} = 1$. If $k = 0$, i.e., α is an empty tuple, we have $\langle \alpha \rangle_n = 2^{[n]}$, consistently with the standard notation. As another example $\langle (0, 1) \rangle_n$ denotes the set of all subsets S such that $n \in S$, but $n - 1 \notin S$.

We shall use the short-hand notation skipping the commas and the brackets, and write $\langle 01 \rangle$ instead of $\langle (0, 1) \rangle$. Furthermore, we shall use the square brackets to encode the repetitions: when α is any tuple of 0's and 1's, $[\alpha]^k$ denotes the tuple obtained by repeating α k times. For example, $0[01]^3$ stands for 0010101 . We shall also use the notation $[\alpha]^*$ to say that α is repeated a certain number of times, without specifying the number of repetitions, which can also be 0. For example,

$$\langle 0[01]^* \rangle = \langle 0 \rangle \cup \langle 001 \rangle \cup \langle 00101 \rangle \cup \dots,$$

and we have $\langle 0[01]^* \rangle_4 = \{0000, 0010, 0100, 0110, 1000, 1010, 1100, 1110, 0001, 1001\}$, while $\langle 1[10]^* \rangle_4 = \{0001, 0011, 0101, 0111, 1001, 1011, 1101, 1111, 0110, 1110\}$.

We let $[\alpha]_{\leq n}^\infty$ denote the tuple obtained by first repeating α infinitely many times, and then truncating it at position n ; it is as much of repeated α as is possible to fit in the first n slots. For example, for $\alpha = 01$, we get $[01]_{\leq 4}^\infty = 0101$ and $[01]_{\leq 3}^\infty = 101$.

11.2. Some useful set decompositions.

In order to define a bijection, which is crucial for the proof of Theorem 1.1, we need a number of specific set decompositions. In the formulations below, we use the symbol $\bar{}$ to denote negation, so $\bar{0} = 1$ and $\bar{1} = 0$.

Lemma 11.2. Whenever $\alpha = (\alpha_1, \dots, \alpha_t)$ is a tuple consisting of 0's and 1's, where $n \geq t$, we have the following decomposition into disjoint subsets:

$$(11.1) \quad 2^{[n]} = [\alpha]_{\leq n}^\infty \cup \bigcup_{i=1}^t \langle \bar{\alpha}_i \alpha_{i+1} \dots \alpha_t [\alpha]^* \rangle.$$

Proof. Take any $S \in 2^{[n]}$, and read its support vector $\chi_S = (a_1, \dots, a_n)$ from right to left, starting with a_n . The first position, where χ_S deviates from $[\alpha]_{\leq n}^\infty$ will show in which of the disjoint sets of the right hand side of (11.1) the set S lies. \square

Corollary 11.3. *For arbitrary $p \geq 1$, we have following identities:*

$$\begin{aligned} 2^{[2p]} &= \langle 0[01]^* \rangle \cup \langle 11[01]^* \rangle \cup [01]^p = \langle 1[10]^* \rangle \cup \langle 00[10]^* \rangle \cup [10]^p \\ 2^{[2p+1]} &= \langle 0[01]^* \rangle \cup \langle 11[01]^* \rangle \cup 1[01]^p = \langle 1[10]^* \rangle \cup \langle 00[10]^* \rangle \cup 0[10]^p \end{aligned}$$

Proof. Follows from Lemma 11.2 by substituting $\alpha = 01$ and $\alpha = 10$ into the equation (11.1) and considering the two cases when n is even or odd. \square

Corollary 11.4. *For any $p \geq 2$ we have the following identities, where all unions on the right hand side are disjoint*

$$(11.2) \quad \langle 0[01]^* \rangle_{2p} = \langle 10[01]^* \rangle \cup \langle 1[10]^*00[01]^* \rangle \cup \langle 00[10]^*00[01]^* \rangle \cup \cup \{[10]^k00[01]^{p-k-1} \mid 0 \leq k \leq p-1\},$$

$$(11.3) \quad \langle 0[01]^* \rangle_{2p+1} = \langle 10[01]^* \rangle \cup \langle 1[10]^*00[01]^* \rangle \cup \langle 00[10]^*00[01]^* \rangle \cup 0[01]^p \cup \cup \{0[10]^k00[01]^{p-k-1} \mid 0 \leq k \leq p-1\}.$$

Proof. Throughout the proof all the unions will be disjoint. We start with the identity

$$(11.4) \quad \langle 0[01]^* \rangle_{2p} = \langle 00[01]^* \rangle_{2p} \cup \langle 10[01]^* \rangle_{2p}.$$

We expand the first term

$$(11.5) \quad \langle 00[01]^* \rangle_{2p} = \langle 00 \rangle_{2p} \cup \langle 0001 \rangle_{2p} \cup \dots \cup \langle 00[01]^k \rangle_{2p} \cup \dots \cup \langle 00[01]^{p-1} \rangle_{2p}.$$

By Corollary 11.3, for all $0 \leq k \leq p-2$ we get

$$(11.6) \quad \langle 00[01]^k \rangle_{2p} = \langle 1[10]^*00[01]^k \rangle_{2p} \cup \langle 00[10]^*00[01]^k \rangle_{2p} \cup [10]^{p-k-1}00[01]^k,$$

and, furthermore, we have $\langle 00[01]^{p-1} \rangle_{2p} = 00[01]^{p-1}$. Taking the union of this equation with the equation (11.6) for all k , we get the identity

$$(11.7) \quad \langle 00[01]^* \rangle_{2p} = \langle 1[10]^*00[01]^* \rangle_{2p} \cup \langle 00[10]^*00[01]^* \rangle_{2p} \cup \cup \{[10]^{p-k-1}00[01]^k \mid 0 \leq k \leq p-1\}.$$

Substituting this into (11.4) proves (11.2).

Proving the odd case (11.3) needs a little modification. The decomposition (11.4) gets replaced with

$$(11.8) \quad \langle 0[01]^* \rangle_{2p+1} = \langle 00[01]^* \rangle_{2p+1} \cup \langle 10[01]^* \rangle_{2p+1} \cup 0[01]^p,$$

while the decomposition (11.6) gets replaced with

$$(11.9) \quad \langle 00[01]^k \rangle_{2p+1} = \langle 1[10]^*00[01]^k \rangle_{2p+1} \cup \langle 00[10]^*00[01]^k \rangle_{2p+1} \cup 0[10]^{p-k-1}00[01]^k,$$

for $0 \leq k \leq p-2$, and $\langle 00[01]^{p-1} \rangle_{2p+1} = \{000[01]^{p-1}, 100[01]^{p-1}\}$. Taking the union over all k , and then substituting back into (11.8) will now yield (11.3). \square

Proposition 11.5. *We have the following identities, where all unions on the right hand side are disjoint*

$$(11.10) \quad \langle 1[10]^* \rangle_{2p} = \langle 11[01]^* \rangle \cup \langle 1[10]^*10[01]^* \rangle \cup \langle 00[10]^*01[01]^* \rangle \cup \cup \{[10]^k[01]^{p-k} \mid 0 \leq k \leq p-1\}, \text{ for all } p \geq 3,$$

$$(11.11) \quad \langle 1[10]^* \rangle_{2p+1} = \langle 11[01]^* \rangle \cup \langle 1[10]^*10[01]^* \rangle \cup \langle 00[10]^*01[01]^* \rangle \cup 1[01]^p \cup \\ \cup \{0[10]^k[01]^{p-k} \mid 0 \leq k \leq p-1\}, \text{ for all } p \geq 2.$$

Proof. Throughout the proof all the unions will be disjoint. We start with proving (11.10). By definition of $[]^*$ -notation, we have the identity

$$(11.12) \quad \langle 1[10]^* \rangle_{2p} = \langle 1 \rangle_{2p} \cup \langle 1[10]^*10 \rangle_{2p}.$$

On the other hand, we have $\langle 1 \rangle_{2p} = \langle 11 \rangle_{2p} \cup \langle 01 \rangle_{2p}$. Substituting this in (11.12) we arrive at

$$(11.13) \quad \langle 1[10]^* \rangle_{2p} = \langle 11 \rangle_{2p} \cup \langle 01 \rangle_{2p} \cup \langle 1[10]^*10 \rangle_{2p}.$$

Next, we shall derive a formula for the term $\langle 01 \rangle_{2p}$. We start with the identity

$$(11.14) \quad 2^{[2p-2]} = \langle 11[01]^* \rangle_{2p-2} \cup \langle 0[01]^* \rangle_{2p-2} \cup [01]^{p-1}$$

which was proved in Corollary 11.3. By definition of $[]^*$, we have

$$(11.15) \quad \langle 0[01]^* \rangle_{2p-2} = \langle 0 \rangle_{2p-2} \cup \langle 001 \rangle_{2p-2} \cup \dots \cup \langle 0[01]^k \rangle_{2p-2} \cup \dots \cup \langle 0[01]^{p-2} \rangle_{2p-2}.$$

Using Corollary 11.3 again, we derive the following identity for all $0 \leq k \leq p-3$

$$(11.16) \quad \langle 0[01]^k \rangle_{2p-2} = \langle 11[01]^*0[01]^k \rangle_{2p-2} \cup \langle 0[01]^*0[01]^k \rangle_{2p-2} \cup 1[01]^{p-k-2}0[01]^k.$$

For future reference, note that

$$\{1[01]^{p-k-2}0[01]^k \mid 0 \leq k \leq p-3\} = \{[10]^k[01]^{p-k-1} \mid 2 \leq k \leq p-1\}.$$

For $k = p-2$ we simply have $\langle 0[01]^{p-2} \rangle_{2p-2} = \{00[01]^{p-2}, 10[01]^{p-2}\}$. Taking the union of that last identity with the identities (11.16), for all $k = 0, \dots, p-3$, we arrive at the formula

$$(11.17) \quad \langle 0[01]^* \rangle_{2p-2} = \langle 11[01]^*0[01]^* \rangle_{2p-2} \cup \langle 0[01]^*0[01]^* \rangle_{2p-2} \cup \\ \cup \{[10]^k[01]^{p-k-1} \mid 1 \leq k \leq p-1\},$$

where the element $00[01]^{p-2}$ went into the second term and the element $10[01]^{p-2}$ went into the third term on the right hand side. We now substitute (11.17) into (11.14) to get the identity

$$2^{[2p-2]} = \langle 11[01]^* \rangle_{2p-2} \cup \langle 11[01]^*0[01]^* \rangle_{2p-2} \cup \langle 0[01]^*0[01]^* \rangle_{2p-2} \cup \\ \cup \{[10]^k[01]^{p-k-1} \mid 0 \leq k \leq p-1\}.$$

Combining this with the suffix 01 we get

$$(11.18) \quad \langle 01 \rangle_{2p} = \langle 11[01]^*01 \rangle_{2p} \cup \langle 11[01]^*0[01]^*01 \rangle_{2p} \cup \langle 0[01]^*0[01]^*01 \rangle_{2p} \cup \\ \cup \{[10]^k[01]^{p-k} \mid 1 \leq k \leq p-1\}.$$

Substituting this into (11.13) we arrive at the formula

$$(11.19) \quad \langle 1[10]^* \rangle_{2p} = \langle 11 \rangle_{2p} \cup \langle 1[10]^*10 \rangle_{2p} \cup \langle 11[01]^*01 \rangle_{2p} \cup \langle 11[01]^*0[01]^*01 \rangle_{2p} \cup \\ \cup \langle 0[01]^*0[01]^*01 \rangle_{2p} \cup \{[10]^k[01]^{p-k} \mid 0 \leq k \leq p-1\}.$$

We now note following identities: $\langle 11 \rangle_{2p} \cup \langle 11[01]^*01 \rangle_{2p} = \langle 11[01]^* \rangle_{2p}$,

$$\langle 1[10]^*10 \rangle_{2p} \cup \langle 11[01]^*0[01]^*01 \rangle_{2p} = \langle 1[10]^*10 \rangle_{2p} \cup \langle 1[10]^*10[01]^*01 \rangle_{2p} = \\ = \langle 1[10]^*10[01]^* \rangle_{2p},$$

and $\langle 0[01]^*0[01]^*01 \rangle_{2p} = \langle 00[10]^*01[01]^* \rangle_{2p}$. Substituting these back into (11.19) will yield (11.10).

No new ideas are needed to show (11.11). All we have to do is retrace the argument used to show (11.10). Throughout the argument, all $\langle \rangle_{2p}$ and $\langle \rangle_{2p-2}$ should be replaced with $\langle \rangle_{2p+1}$ and $\langle \rangle_{2p-1}$. Then (11.12) and (11.13) remain the same, subject to the subscript change we just mentioned, while (11.14) gets replaced with

$$(11.20) \quad 2^{[2p-1]} = \langle 11[01]^* \rangle_{2p-1} \cup \langle 0[01]^* \rangle_{2p-1} \cup 1[01]^{p-1}.$$

The identity (11.15) becomes

$$\langle 0[01]^* \rangle_{2p-1} = \langle 0 \rangle_{2p-1} \cup \langle 001 \rangle_{2p-1} \cup \cdots \cup \langle 0[01]^{p-2} \rangle_{2p-1} \cup \langle 0[01]^{p-1} \rangle_{2p-1},$$

and we get

$$\langle 0[01]^k \rangle_{2p-1} = \langle 11[01]^* 0[01]^k \rangle_{2p-1} \cup \langle 0[01]^* 0[01]^k \rangle_{2p-1} \cup [01]^{p-k-1} 0[01]^k,$$

for $0 \leq k \leq p-2$, and $\langle 0[01]^{p-1} \rangle_{2p-1} = \{0[01]^{p-1}\}$. The identity (11.17) becomes

$$(11.21) \quad \langle 0[01]^* \rangle_{2p-1} = \langle 11[01]^* 0[01]^* \rangle_{2p-1} \cup \langle 0[01]^* 0[01]^* \rangle_{2p-1} \cup \{0[10]^k [01]^{p-k-1} \mid 0 \leq k \leq p-1\}.$$

Substituting this into (11.20) yields

$$2^{[2p-1]} = \langle 11[01]^* \rangle_{2p-1} \cup 1[01]^{p-1} \cup \langle 11[01]^* 0[01]^* \rangle_{2p-1} \cup \langle 0[01]^* 0[01]^* \rangle_{2p-1} \cup \{0[10]^k [01]^{p-k-1} \mid 0 \leq k \leq p-1\}.$$

and combining with the suffix 01 as above, and then substituting it into the analog of (11.13) gives the analog of the formula (11.19), which now says the following

$$(11.22) \quad \langle 1[10]^* \rangle_{2p+1} = \langle 11 \rangle_{2p+1} \cup \langle 1[10]^* 10 \rangle_{2p+1} \cup \langle 11[01]^* 01 \rangle_{2p+1} \cup 1[01]^p \cup \langle 11[01]^* 0[01]^* 01 \rangle_{2p+1} \cup \langle 0[01]^* 0[01]^* 01 \rangle_{2p+1} \cup \{0[10]^k [01]^{p-k} \mid 0 \leq k \leq p-1\}.$$

Repeating the same transformations as in the $2p$ -case we derive (11.11). \square

We have now used an explicit constructive argument to prove the Proposition 11.5. We feel, it is well-suited for explaining the inner mechanics of the formulae (11.10) and (11.11). However, it is also possible to give a much shorter implicit argument. To save space, we restrict ourselves to giving a sketch of how to show (11.10) in two steps. Step 1: count the number of elements on both side and derive that they are both equal to $\frac{2}{3}(4^p - 1)$. Note that we do not know yet that the sets on the right hand side are disjoint, so overlaps are counted multiple times. Step 2: show that every element from the left hand side can be found on the right hand side. This can be done by considering several different cases. Once this is done we know both that the two sides are equal and that the sets on the right hand side are disjoint. The identity (11.11) can be shown exactly the same, with the number of elements on both sides being equal to $\frac{1}{3}(4^p - 1)$.

11.3. The main bijection and the proof of our main theorem.

We let $\Sigma\langle\alpha\rangle$ denote the set of all subsets from a family Σ which end on α , in other words $\Sigma\langle\alpha\rangle := \Sigma \cap \langle\alpha\rangle$. Clearly, all the decompositions above can be intersected with Σ . This will give a number of different decompositions of Σ , such as

$$\Sigma = \Sigma\langle 0[01]^* \rangle \cup \Sigma\langle 11[01]^* \rangle \cup \{[01]^{n/2}\}$$

Proposition 11.6. *For any integer $n \geq 5$, and any $0 \leq t \leq n-1$, there exists a bijection*

$$\Phi_t^n : C_t^n\langle 0[01]^* \rangle \rightarrow C_{t+1}^n\langle 1[10]^* \rangle,$$

such that for all $S \in \langle 0[01]^ \rangle_n$, we have $S \subseteq \Phi_t^n(S)$.*

Proof. Assume first that $n = 2p$ is even, and compare the formula (11.2) with (11.10). We see a strong similarity, and define the bijection Φ_t^n by the rules

$$\begin{aligned} \alpha 10[01]^k &\mapsto \alpha 11[01]^k, \\ \alpha 1[10]^m 00[01]^k &\mapsto \alpha 1[10]^m 10[01]^k, \\ \alpha 00[10]^m 00[01]^k &\mapsto \alpha 00[10]^m 01[01]^k, \\ [10]^k 00[01]^{p-k-1} &\mapsto [10]^k 01[01]^{p-k-1}, \end{aligned}$$

for all $k, m \geq 0$, and all strings α . When $n = 2p + 1$ is odd, we compare the formula (11.3) with (11.11) instead, and the last rule of the bijection gets changed to

$$\begin{aligned} 0[10]^k 00[01]^{p-k-1} &\mapsto 0[10]^k 01[01]^{p-k-1}, \\ 0[01]^p &\mapsto 1[01]^p, \end{aligned}$$

for all k . □

Definition 11.7. Assume $2 \leq t \leq n$, and consider bijections

$$\begin{aligned} \gamma : C_t^n \langle 11[01]^* \rangle &\rightarrow C_{t-1}^{n-1} \langle 1[10]^* \rangle & \rho : C_{t-2}^{n-1} \langle 0[01]^* \rangle &\rightarrow C_{t-2}^n \langle 00[10]^* \rangle \\ \alpha 1[10]^k 1 &\mapsto \alpha 1[10]^k & \alpha 0[01]^* &\mapsto \alpha 0[01]^* 0 \end{aligned}$$

where α is any string, and k is any positive integer.² We then define a bijection

$$\Psi_t^n : C_t^n \langle 11[01]^* \rangle \rightarrow C_{t-2}^n \langle 00[10]^* \rangle$$

as a composition $\Psi_t^n := \rho \circ (\Phi_{t-2}^{n-1})^{-1} \circ \gamma$.

Let $M_r^n = \{S \subseteq [n] \mid |S| \equiv r \pmod{3}\}$.

Proposition 11.8. For all $t \geq 1$, there exists a bijection

$$\Lambda : M_0^{6t} \setminus [01]^{3t} \rightarrow M_1^{6t},$$

such that either $S \subset \Lambda(S)$ or $\Lambda(S) \subset S$. Under this bijection we have $\Lambda([1]^{6t}) = [1]^{6t-2}00$.

Proof. By Corollary 11.3 we have

$$\begin{aligned} M_0^{6t} \setminus [01]^{3t} &= \bigcup_{k=0}^{2t-1} C_{3k}^n \langle 0[01]^* \rangle \cup \bigcup_{k=1}^{2t} C_{3k}^n \langle 11[01]^* \rangle, \\ M_1^{6t} &= \bigcup_{k=0}^{2t-1} C_{3k+1}^n \langle 1[10]^* \rangle \cup \bigcup_{k=1}^{2t} C_{3k-2}^n \langle 00[10]^* \rangle, \end{aligned}$$

where all the unions are disjoint. We now define Λ by saying that the restriction of Λ to $C_{3k}^n \langle 0[01]^* \rangle$ is equal to Φ_{3k}^n , and the restriction of Λ to $C_{3k}^n \langle 11[01]^* \rangle$ is equal to Ψ_{3k}^n . By what is proved until now, this is clearly a bijection. □

We are now ready to prove our main theorem.

Proof of Theorem 1.1. We have a matching where the only unmatched sets are $[01]^{3t}$ and $[1]^{6t-2}00$. We fix that by using an augmenting path

$$[01]^{3t} \rightsquigarrow [01]^{3t-1}11 \rightarrow [01]^{3t-1}10 \rightsquigarrow [01]^{3t-2}1110 \rightarrow [01]^{3t-2}1100 \rightsquigarrow [1]^{6t-2}00,$$

where the edges $[01]^{3t-1}11 \rightarrow [01]^{3t-1}10$ and $[01]^{3t-2}1110 \rightarrow [01]^{3t-2}1100$ are matching edges. □

²Note how we use the facts that $11[01]^k = 1[10]^k 1$ and $00[10]^k = 0[01]^k 0$.

Distributed Computing Context 11.9. *Theorem 1.1 means that whenever the number of processes is divisible by 6, the Weak Symmetry Breaking task can be solved in 3 rounds. In particular, there are infinitely many values for the number of processes, for which this task can be solved using a constant number of rounds.*

The smallest values of n which are not covered by Theorem 1.1 are $n = 10, 14, 15$. The binomial Diophantine equations associated to $n = 10$ and to $n = 14$ do not have primitive solutions. For $n = 15$ we do find several primitive solutions, for example $x_1 = x_3 = x_5 = x_{10} = 1$, $x_4 = x_6 = x_{13} = -1$, and for all other i we take $x_i = 0$. This corresponds to the following arithmetic identity:

$$\binom{15}{1} + \binom{15}{3} + \binom{15}{5} + \binom{15}{10} = \binom{15}{0} + \binom{15}{4} + \binom{15}{6} + \binom{15}{13} = 6476.$$

A computer search can then be used to find a comparable matching between disjoint cardinal proper families of subsets of $[15]$, implying $\text{sb}(15) \leq 3$.

Proposition 11.10. *The binomial Diophantine equation associated to n has solutions if and only if n is not a prime power. Furthermore, there are infinitely many values of n , say $n = 6t$, for arbitrary natural number m , for which the binomial Diophantine equation associated to n has a primitive solution.*

Proof. If $n = p^m$, then all the binomial coefficients $\binom{p}{1}, \dots, \binom{p}{p-1}$ are divisible by p , so obviously the binomial Diophantine equation associated to n has no solutions. Otherwise, the greatest common divisor of these coefficients is equal to 1, and so a solution can be found by Euclidean algorithm.

For $n = 6t$ we have an identity

$$(11.23) \quad \binom{6t}{0} + \binom{6t}{3} + \binom{6t}{6} + \dots + \binom{6t}{6t-3} = \binom{6t}{1} + \binom{6t}{4} + \binom{6t}{7} + \dots + \binom{6t}{6t-2},$$

so the following is a primitive solution: $x_1 = x_4 = \dots = x_{6t-2} = 1$, $x_3 = x_6 = \dots = x_{6t-3} = -1$, and all other coefficients are equal to 0. \square

11.4. Example $t = 1$.

When $t = 1$ we are dealing with the subsets of the set $[6]$. We have $|M_0^6 \setminus [01]^3| = |M_1^6| = 21$ and we need to match the elements of these two sets with each other. To start with we have $C_0^6 \langle 0[01]^* \rangle = 000000$, $C_1^6 \langle 1[10]^* \rangle = 000001$, furthermore $000000 \in C_0^6 \langle 00[10]^* 00[01]^* \rangle$, and hence $\Phi_0^6(000000) = 000001$. Similarly, $C_6^6 \langle 11[01]^* \rangle = 111111$, $C_4^6 \langle 00[10]^* \rangle = 111100$, and $\Psi_6^6(111111) = 111100$. It remains to mutually match the 14-element sets $C_3^6 \langle 0[01]^* \rangle$ and $C_4^6 \langle 1[10]^* \rangle$, and the 5-element sets $C_3^6 \langle 11[01]^* \rangle$ and $C_1^6 \langle 00[10]^* \rangle$.

We start with the two 14-element sets. In this case the formula (11.2) simplifies to

$$C_3^6 \langle 0[01]^* \rangle = C_3^6 \langle 10[01]^* \rangle \cup C_3^6 \langle 1[10]^* 00[01]^* \rangle,$$

where the first set in the union has 9 elements, and the second one has 5 elements. Similarly, the formula (11.10) simplifies to

$$C_4^6 \langle 1[10]^* \rangle = C_4^6 \langle 11[01]^* \rangle \cup C_4^6 \langle 1[10]^* 10[01]^* \rangle.$$

Our matching rule is now the following

$C_3^6 \langle 10[01]^* \rangle$		$C_4^6 \langle 11[01]^* \rangle$		$C_3^6 \langle 1[10]^* 00[01]^* \rangle$		$C_4^6 \langle 1[10]^* 10[01]^* \rangle$
100101	\mapsto	110101		110100	\mapsto	110110
011001	\mapsto	011101		011100	\mapsto	011110
101001	\mapsto	101101		101100	\mapsto	101110
001110	\mapsto	001111		111000	\mapsto	111010
010110	\mapsto	010111		110001	\mapsto	111001
100110	\mapsto	100111				
011010	\mapsto	011011				
101010	\mapsto	101011				
110010	\mapsto	110011				

For the above-mentioned 5-element sets we need to use the bijection Ψ_3^6 , whose definition is somewhat more complicated. The composition from Definition 11.7 yields in our case the following maps:

$$\begin{array}{ccccccc}
100011 & \longrightarrow & 10001 & \longrightarrow & 10000 & \longrightarrow & 100000 \\
010011 & \longrightarrow & 01001 & \longrightarrow & 01000 & \longrightarrow & 010000 \\
001101 & \longrightarrow & 00110 & \longrightarrow & 00100 & \longrightarrow & 001000 \\
000111 & \longrightarrow & 00011 & \longrightarrow & 00010 & \longrightarrow & 000100 \\
001011 & \longrightarrow & 00101 & \longrightarrow & 00001 & \longrightarrow & 000010
\end{array}$$

Finally, we need to alter our matching one time since 010101 and 111100 are not matched, alternatively, we can think that 111100 is matched to 111111, which needs the same modification. This is done as is described in the proof of Theorem 1.1. We break down the bonds $010110 \mapsto 010111$ and $011100 \mapsto 011110$, and take the following 3 edges as the new matching edges: $010101 \mapsto 010111$, $010110 \mapsto 011110$, and $011100 \mapsto 111100$.

12. CURRENT BOUNDS FOR THE SYMMETRY BREAKING NUMBER

We strongly believe that the techniques developed in this paper can be extended to deal with many other values of n . This has recently been confirmed as follows.

Definition 12.1. [Ko16, Definitions 1.1, 1.2]. Assume that n is a natural number and that for some numbers $0 \leq a_1 < \dots < a_k \leq n$ and $0 \leq b_1 < \dots < b_m \leq n$, we have an equality

$$(12.1) \quad \binom{n}{a_1} + \dots + \binom{n}{a_k} = \binom{n}{b_1} + \dots + \binom{n}{b_m}.$$

We call such an identity a **binomial identity**.

Let Σ , resp. Λ , be set of all subsets of $[n]$, with cardinalities a_1, \dots, a_k , resp. b_1, \dots, b_m . We say that the binomial identity (12.1) is **orderable** if there exists a bijection $\Phi : \Sigma \rightarrow \Lambda$, such that for all $S \in \Sigma$ we either have $S \subseteq \Phi(S)$ or $S \supseteq \Phi(S)$.

In this paper we have constructed a complicated explicit bijection for the case $n = 6t$, $k = m = 2t$, $\{a_1, \dots, a_{2t}\} = \{0, 3, \dots, 6t - 3\}$, and $\{b_1, \dots, b_{2t}\} = \{1, 4, \dots, 6t - 2\}$, see Section 11. It would be interesting to see whether a simpler bijection can be found.

Recently, we proved the following combinatorial theorem.

Theorem 12.2. [Ko16, Theorem 1.3]. All binomial identities are orderable.

It has been shown in [Ko16] that, together with Theorem 10.9, this implies the following bound on the symmetry breaking number.

Theorem 12.3. [Ko16, Theorem 3.2]. *Assume the binomial Diophantine equation associated to n has a primitive solution, then we have $sb(n) \leq 3$.*

Our current knowledge about $sb(n)$ is summarized in the Table 12.1.

Bound	Source
$sb(n) = \infty$ if and only if n is a prime power	[CR08, CR10, CR12a]
$sb(n) = O(n^{q+3})$, if n is not a prime power and q is the largest prime power in the prime factorization of n	[ACHP13]
$sb(n) \geq 2$	[Ko15a]
$sb(6t) \leq 3$, for all $t \geq 1$	Theorem 1.1 above; the case $t = 1$ in [Ko15a]
$sb(n) \leq 3$, if the binomial Diophantine equation associated to n has a primitive solution	[Ko16]

TABLE 12.1. The known bounds of $sb(n)$.

In general, we feel that the work presented in this paper is suggesting that we need to change our paradigm. When looking for lower bounds for the complexity of the distributed protocols solving Weak Symmetry Breaking, the focus needs to shift from the number of processes n itself to the sizes of the coefficients in the solution of the binomial Diophantine equation associated to n .

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13. APPENDIX: PATH BUILDING KIT

In this appendix we list different edge paths in Γ_n which are used elsewhere in the paper. These paths are alternating with respect to some give matching μ , and we use $\xrightarrow{\mu}$ to denote edges belonging to the matching, while \rightsquigarrow denotes all other edges.

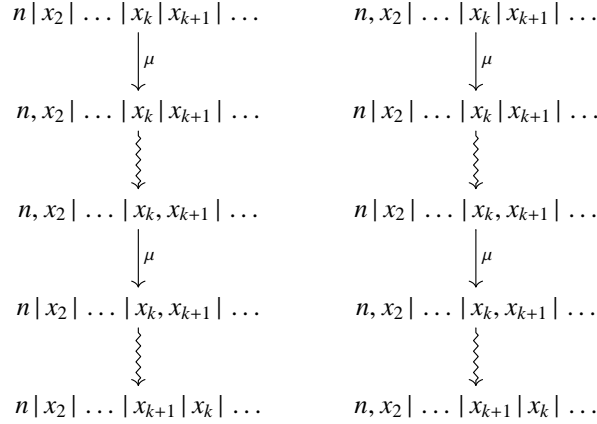


FIGURE 13.1. Paths swap_k^I and swap_k^{II} , for $3 \leq k \leq n-1$.

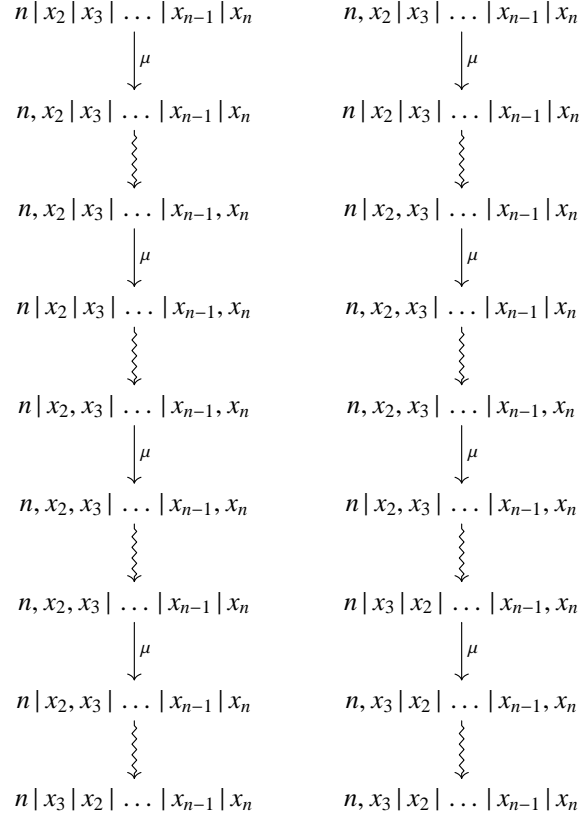
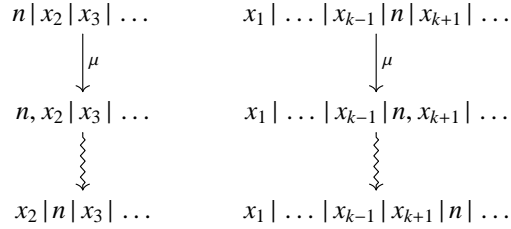
FIGURE 13.2. Paths swap_2^I and swap_2^{II} .

FIGURE 13.3. On the left hand side we have the alternating path up_1^I , which is legal if either $x_2 \notin V$ or if $x_2 \in \Omega$. On the right hand side we have alternating path up_k^I , for $2 \leq k \leq n-1$, which is legal if either $x_1 \notin V$, or if $x_1 | \dots | x_{k-1} \in \Omega$ and $x_1 | \dots | x_{k-1} | x_{k+1} \in \Omega$.

$$\begin{array}{c}
x_1, x_2 | x_3 | \dots | x_{k-1} | n | x_{k+1} | \dots \\
\downarrow \mu \\
x_1, x_2 | x_3 | \dots | x_{k-1} | x_{k+1}, n | \dots \\
\downarrow \text{zigzag} \\
x_1, x_2 | x_3 | \dots | x_{k-1} | x_{k+1} | n | \dots
\end{array}$$

FIGURE 13.4. Path up_k^{II} , for $3 \leq k \leq n-1$: legal if either $\{x_1, x_2\} \not\subseteq V$, or if $\{x_1, x_2\} | x_3 | \dots | x_{k-1} \in \Omega$ and $\{x_1, x_2\} | x_3 | \dots | x_{k-1} | x_{k+1} \in \Omega$.

$$\begin{array}{cc}
n, a_2 | n-1 | a_4 | \dots & n-1, n | a_3 | a_4 | \dots \\
\downarrow \mu & \downarrow \mu \\
n | a_2 | n-1 | a_4 | \dots & n | n-1 | a_3 | a_4 | \dots \\
\downarrow \text{zigzag} & \downarrow \text{zigzag} \\
n | a_2, n-1 | a_4 | \dots & n | n-1 | a_3, a_4 | \dots \\
\downarrow \mu & \downarrow \mu \\
n, a_2, n-1 | a_4 | \dots & n-1, n | a_3, a_4 | \dots \\
\downarrow \text{zigzag} & \downarrow \text{zigzag} \\
n-1 | n, a_2 | a_4 | \dots & n-1 | n | a_3, a_4 | \dots \\
\downarrow \mu & \downarrow \mu \\
n-1 | n | a_2 | a_4 | \dots & n-1 | n, a_3, a_4 | \dots \\
\downarrow \text{zigzag} & \downarrow \text{zigzag} \\
n-1 | n | a_2, a_4 | \dots & n-1 | a_3 | n, a_4 | \dots \\
\downarrow \mu & \downarrow \mu \\
n-1 | n, a_2, a_4 | \dots & n-1 | a_3 | n | a_4 | \dots \\
\downarrow \text{zigzag} & \downarrow \text{zigzag} \\
n-1 | a_2 | n, a_4 | \dots & n-1, a_3 | n | a_4 | \dots \\
\downarrow \mu & \\
n-1 | a_2 | n | a_4 | \dots & \\
\downarrow \text{zigzag} & \\
n-1, a_2 | n | a_4 | \dots &
\end{array}$$

FIGURE 13.5. Paths $\text{specup}_2^{\text{II}}$ and up_2^{II} .

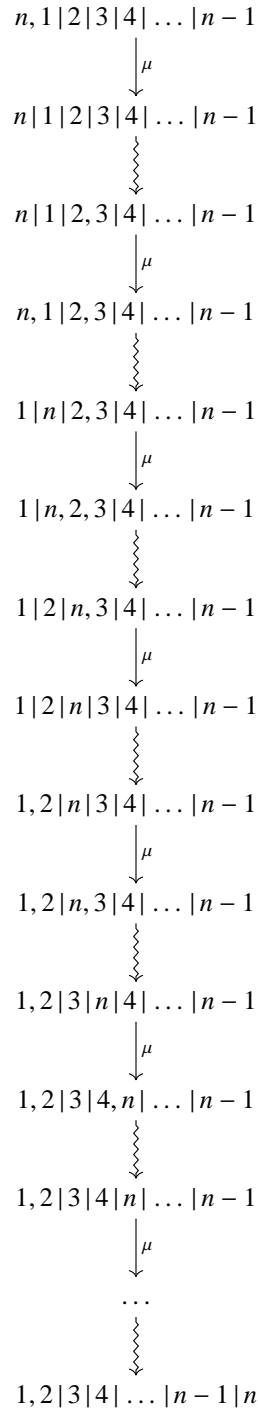


FIGURE 13.6. The final part for the path in Lemma 6.9.

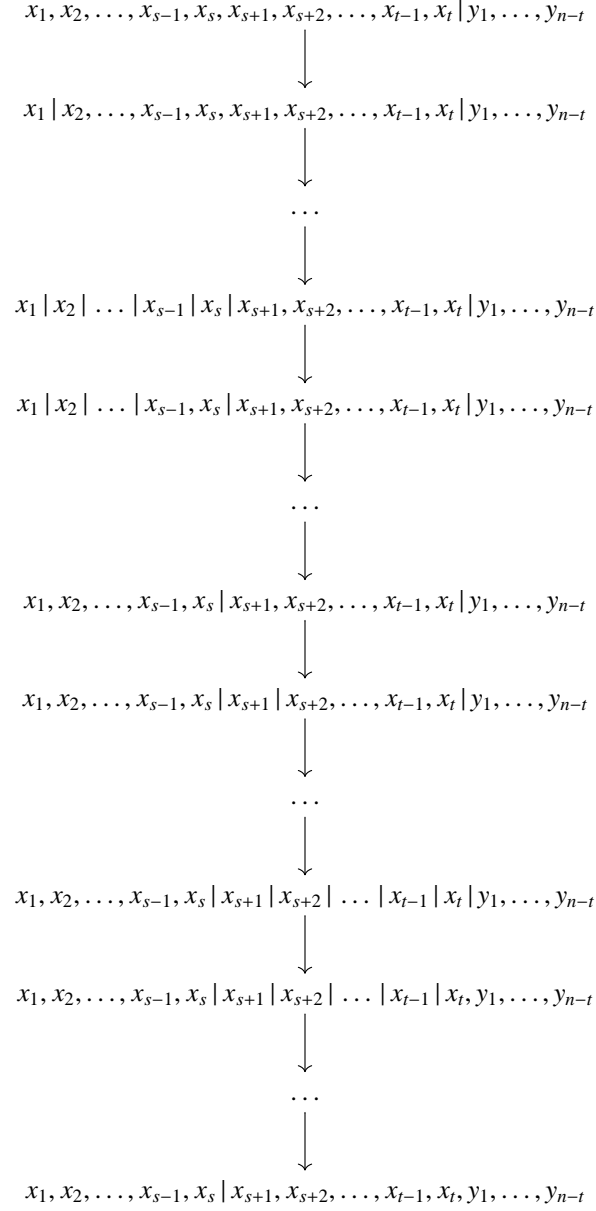


FIGURE 13.7. The standard path $p_{S,T}$ for $S = (x_1, \dots, x_s)$, $T = (x_1, \dots, x_t)$.